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ON THE RESONANCE CONCEPT IN SYSTEMS OF LINEAR AND NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

by Rahmi Ibrahim Ibrahim Abdel Karim

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Doctoral Dissertation, Tanta, Egypt, 1963.

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ON THE RESONANCE CONCEPT IN SYSTEMS OF LINEAR AND NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

By Rahmi Ibrahim Ibrahim Abdel Karim

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ON THE RESONANCE CONCEPT IN SYSTEMS OF LINEAR AND NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

From the Natural Science Philosophical Faculty of the Carolo-Wilhelmina Polytechnic Institute, Braunschweig

As dissertation for the degree of a Doctor of Natural Sciences

(Dr.rer.nat.)

Rahmi Ibrahim Ibrahim Abdel Karim
of Tanta/Egypt United Arab Republic

Submitted: 19/12/1962

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Reporter: Prof. Dr.R.Iglisch

Coreporter: Prof. Dr.H.-J.Kowalskiy

ON THE RESONANCE CONCEPT IN SYSTEMS OF LINEAR AND NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Rahmi Ibrahim Ibrahim Abdel Karim

Theorems on the resonance cases for linear and nonlinear ordinary differential equations of the first to the n^{th} order are derived and proved in detail, using an earlier report by the same author as partial basis. Minimal orders of magnitude of the solutions and their derivatives are given and methods for the formation of examples, with sample calculations in matrix notation, are described.

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PART I

THE RESONANCE CONCEPT IN SYSTEMS OF n LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Section 1. Problem Formulation, Principal Results

R.Iglisch investigated the resonance concept in linear ordinary differential equations of the second order (Bibl.5). These considerations will be extended here to systems of n ordinary linear differential equations of the first order which will be written in matrix form with $\mathfrak{x}(t)$ as the sought vector:

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{A}(t)\,\mathfrak{x} + \mathfrak{f}(t); \tag{1}$$

where the square (for example, real) matrix $\mathfrak{A}(t)$ is to be continuous in t and periodic with the period P, i.e., all n^2 elements $a_{ik}(t)$ are continuous functions in t, periodic with P; the (for example, also real) vector $\mathfrak{f}(t)$ is assumed as also being continuous and periodic with P:

$$\mathfrak{A}(t+P) = \mathfrak{A}(t); \quad \mathfrak{f}(t+P) = \mathfrak{f}(t). \tag{2}$$

The homogeneous system conjugate to eq.(1) reads

$$\frac{d\mathfrak{y}}{dt} = \mathfrak{A}(t)\,\mathfrak{y} \tag{3}$$

while the "adjoint" system is

$$\frac{d\mathfrak{z}}{dt} = -\mathfrak{A}^T(t)\,\mathfrak{z}, \tag{4}$$

where the superscript T is to denote the transition to the transposed matrix.

Definition 1. In the inhomogeneous differential equation system (1), the resonance case is present if the adjoint system (4) has at least one solution vector $\mathfrak{z}(t)$ periodic with P, for which

$$\int_0^P \mathfrak{z}^T(t)\,\mathfrak{f}(t)\,dt = C \neq 0 \tag{5}$$

is valid.

In Section 4, we will prove:

Theorem 1. In the resonance case, any solution vector $\mathfrak{x}(t)$ of eq.(1) assumes arbitrarily large values, increasing without bounds with increasing t.

Definition 2. In eq.(1), we have the principal case if eq.(4) has no /8 solution vector $\mathfrak{z}(t)$ periodic with P.

In Section 5, we will prove:

Theorem 2. In the principal case, solutions $\mathfrak{x}(t)$ of eq.(1), remaining restricted for all t, are in existence, for example, the uniquely existing solution periodic with P.

Definition 3. Equation (1) represents the exceptional case if eq.(4) does have solution vectors $\mathfrak{z}(t)$ periodic with P but if the following relation applies to all these solutions periodic with P:

$$\int_0^P \mathfrak{z}^T(t)\,\mathfrak{f}(t)\,dt = 0. \tag{6}$$

In Section 6, we will prove:

Theorem 3. Even in the exceptional case, restricted solutions $\mathfrak{x}(t)$ of eq.(1) exist for all values of t, for example, solutions periodic with P but no longer uniquely defined.

Sections 2 and 3 contain auxiliary considerations on the inhomogeneous system (1), specifically on the correlation between the homogeneous system (3) and the adjoint system (4). In this case, the periodicity stipulation (2) will be introduced only in Section 3.

Many of the results are already known, but they are here derived in a manner that requires no specialized knowledge.

Section 2. Auxiliary Considerations on Systems of Linear Differential Equations of the First Order

In this Section, the periodicity stipulation (2) will not be used.

Let $\eta_1(t)$, ..., $\eta_n(t)$ be a linear independent solution system (fundamental system) of eq.(3) at the point t_0, which can be combined into the solution matrix

$$\mathfrak{Y}(t) = (\eta_1(t), \ldots, \eta_n(t)) \tag{7}$$

We then have

$$\operatorname{Det} \mathfrak{Y}(t_0) \neq 0. \tag{8}$$

Theorem 4. From eq.(8) it follows that

$$Y(t) = \operatorname{Det} \mathfrak{Y}(t) \neq 0 \tag{9}$$

for all values of t.

Proof. If the conjugates [subdeterminants with correct sign $(-1)^{i+\nu}$] for the element $y_{i\nu}(t)$ in eqs.(7) or (9) are denoted by $Y_{i\nu}(t)$ and putting

$$\delta_{ik} = \begin{cases} 0 & \text{for } i \neq k, \\ 1 & \text{for } i = k, \end{cases} \tag{10}$$

the following result will be obtained, taking eq.(3) into consideration and denoting the differentiation to t by a prime:

$$\frac{dY}{dt} = \sum_{\nu,i} y'_{i\nu} Y_{i\nu} = \sum_{\nu,i,k} a_{ik} y_{k\nu} Y_{i\nu} = \sum_{i,k} a_{ik} \delta_{ki} Y = Y \sum_{i} a_{ii}$$

and thus

$$Y(t) = Y(t_0) \exp\left(\int_{t_0}^{t} (a_{11} + a_{22} + \cdots + a_{nn})\, dt \right).$$

From this, theorem 4 follows directly.

Theorem 5. In addition to eq.(7), the expression

$$\mathfrak{Y}^*(t) = \mathfrak{Y}(t) \cdot \mathfrak{C} \tag{11}$$

at an arbitrary constant matrix $\mathfrak{C}$ with a determinant C differing from zero, represents a fundamental system of solutions of eq.(3).

Proof. Equations (7) and (3), together with

$$\mathfrak{Y}' = (\mathfrak{y}_1', \mathfrak{y}_2', \ldots, \mathfrak{y}_n') \tag{12}$$

can be combined into

$$\frac{d\mathfrak{Y}}{dt} = \mathfrak{A}(t)\,\mathfrak{Y}. \tag{13}$$

From this, we can calculate

$$(\mathfrak{Y}\mathfrak{C})' = \mathfrak{Y}'\mathfrak{C} = \mathfrak{A}\,\mathfrak{Y}\mathfrak{C} = \mathfrak{A}(\mathfrak{Y}\mathfrak{C}).$$

This means that each $\mathfrak{Y}\mathfrak{C}$ is a solution system of eq.(3).

That we also have $\det(\mathfrak{Y}\mathfrak{C}) \neq 0$, follows from

$$\det(\mathfrak{Y}\mathfrak{C}) = Y \cdot C \neq 0.$$

Theorem 6. The vectors

$$\mathfrak{z}_i(t) = \frac{1}{Y(t)}\begin{pmatrix} Y_{1i}(t) \\ Y_{2i}(t) \\ \vdots \\ Y_{ni}(t) \end{pmatrix}, \quad i = 1, \ldots, n, \tag{14}$$

which can be combined into the matrix

$$\mathfrak{Z}(t) = \left(\frac{1}{Y(t)} \cdot Y_{ki}(t)\right) \tag{15}$$

form a fundamental system of solutions of eq.(4).

Proof. Equation (15) indicates directly that

$$\mathfrak{Z}^T \cdot \mathfrak{Y} = \mathfrak{E}, \tag{16}$$

i.e., that it is the unit matrix. On introducing the reciprocal matrix $\mathfrak{Y}^{-1}$, we can then also write

$$\mathfrak{Z}^T = \mathfrak{Y}^{-1}, \quad \mathfrak{Z} = (\mathfrak{Y}^{-1})^T. \tag{17}$$

From eqs.(16) or (17) it follows directly that

$$Z = \det \mathfrak{Z} \neq 0.$$

The statement that $\mathfrak{Z}$ is the solution matrix of eq.(4) is derived as follows:

According to eq.(16), we have

$$\mathfrak{Z}^T \cdot \mathfrak{Y} + \mathfrak{Z}^T \mathfrak{Y}' = 0.$$

i.e., with eq.(13) /10

$$-\mathfrak{Z}^{T\cdot}\mathfrak{Y}=\mathfrak{Z}^{T}\mathfrak{A}\,\mathfrak{Y}.\quad -\mathfrak{Z}^{T\prime}=\mathfrak{Z}^{T}\mathfrak{A}\,\mathfrak{Y}\,\mathfrak{Y}^{-1}=\mathfrak{Z}^{T}\mathfrak{A},$$

so that

$$\mathfrak{Z}'=-\mathfrak{A}^{T}\mathfrak{Z}. \tag{18}$$

which coincides with eq.(4).

Thus, theorems 5 and 6 yield directly:

Theorem 7. A general fundamental system of solutions of eq.(4) is represented by the matrix

$$\mathfrak{Z}^{*}=\mathfrak{Z}\cdot\mathfrak{C}=(\mathfrak{Y}^{-1})^{T}\cdot\mathfrak{C} \tag{19}$$

at arbitrary constant matrix $\mathfrak{C}$ with a determinant C differing from zero.

Denoting

$$L(\eta)\equiv\frac{d\eta}{dt}-\mathfrak{A}(t)\,\eta \tag{20}$$

and

$$L^{*}(\zeta)\equiv-\frac{d\zeta}{dt}-\mathfrak{A}^{T}(t)\,\zeta, \tag{21}$$

the following is valid for two arbitrary vectors $\eta(t)$ and $\zeta(t)$:

$$\zeta^{T}L(\eta)-(L^{*}(\zeta))^{T}\cdot\eta=\frac{d}{dt}(\zeta^{T}\cdot\eta) \tag{22}$$

and

$$(L(\eta))^{T}\cdot\zeta-\eta^{T}L^{*}(\zeta)=\frac{d}{dt}(\eta^{T}\cdot\zeta). \tag{23}$$

Therefore, the (expanded) Lagrange identity

$$\frac{d}{dt}(\zeta^{T}\cdot\eta)=0=\frac{d}{dt}(\eta^{T}\cdot\zeta). \tag{24}$$

is valid for the solution vectors η and ζ of eqs.(3) and (4), respectively.

Analogously, the following is valid for the solution vectors of eqs.(1) and (4):

$$\frac{d}{dt}(\zeta^{T}\cdot\xi)=\zeta^{T}f,\quad \frac{d}{dt}(\xi^{T}\cdot\zeta)=f^{T}\cdot\zeta. \tag{25}$$

Section 3. Systems of Linear Differential Equations with Periodic Coefficients

In this Section, the periodicity stipulation (2) is essential.

Theorem 8. The homogeneous systems (3) and (4) have the same number of ρ

linearly independent solutions periodic with P ($0 \leq \rho \leq n$).

Proof*. Let ρ be the number of linearly independent solutions of eq.(3), periodic with P. It is merely necessary to demonstrate that eq.(4) has exactly ρ linearly independent solutions that are periodic with P, since the conclusion can be reversed in view of the fact that eq.(3) is the adjoint system to eq.(4).

Case 1. Let $\rho = n$. Here, the entire matrix $\mathfrak{A}(t)$ is periodic with P so that also the matrix $\mathfrak{Z}(t)$ according to eq.(15) will be periodic.

Case 2. Let $0 < \rho < n$. Assume that the $\eta_\nu(t)$, with $\nu = 1, 2, \ldots, \rho$, /11 are periodic with P. Then, the following is valid for any arbitrary point t_1:

$$\eta_\mu(t)\Big|_{t_1}^{t_1+P} = \eta_\mu(t_1 + P) - \eta_\mu(t_1) \neq 0 \quad \text{for} \quad \mu = \rho + 1, \ldots, n. \tag{26}$$

If the equal sign were present in eq.(26), the periodicity of this $\eta_\mu(t)$ with the period P would follow from eq.(3) with consideration of eq.(2). In the following, let t_1 be an arbitrarily selected but then retained point. Integration of the first equation in the system (24) over t_1 and $t_1 + P$ will yield, for each $k = 1, 2, \ldots, n$,

$$\mathfrak{Z}_k^T(t)\Big|_{t_1}^{t_1+P}\eta_\nu(t_1) = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, \rho. \tag{27}$$

These are n linear homogeneous equations which have at least the ρ linearly independent solutions $\eta_\nu(t_1)$ for $\nu = 1, 2, \ldots, \rho$. It will be demonstrated that no further solution vector $\mathbf{u}_1$, linearly independent of this, can exist. Conversely, let us assume that, for such a $\mathbf{u}_1$, the following is also valid

$$\mathfrak{Z}_k^T(t)\Big|_{t_1}^{t_1+P}\mathbf{u}_1 = 0 \tag{28}$$

Then, let us define a vector $\mathbf{u}(t)$ from eq.(3):

$$\frac{d\mathbf{u}}{dt} = \mathfrak{A}(t)\,\mathbf{u} \quad \text{with} \quad \mathbf{u}(t_1) = \mathbf{u}_1. \tag{29}$$

---

* By R.Iglisch.

For this vector $u(t)$, the first equation of the system (24), by integration over t_1 and $t_1 + P$ for $k = 1, 2, \ldots, n$, will yield

$$\delta_k^T(t_1 + P)\,u(t_1 + P) - \delta_k^T(t_1)\,u(t_1) = 0$$

and, after subtraction of eq.(28),

$$\delta_k^T(t_1 + P)\,[u(t_1 + P) - u(t_1)] = 0.$$

This represents a linear homogeneous system of equations with a determinant differing from zero; consequently,

$$u(t_1 + P) = u(t_1)$$

must be valid from which, because of eqs.(29) and (2), the periodicity of $u(t)$ with P is obtained. Consequently, because of eq.(26), $u_1 = u(t_1)$ is linearly dependent on $h_1(t_1), \ldots, h_0(t_1)$.

Thus, we know that the system of equations (27) has exactly ρ linearly independent solutions. Therefore, the matrix

$$\mathfrak{Z}(t)\Big|_{t_1}^{t_1+P} = \left(\delta_1(t)\Big|_{t_1}^{t_1+P}, \delta_2(t)\Big|_{t_1}^{t_1+P}, \ldots, \delta_n(t)\Big|_{t_1}^{t_1+P}\right)$$

has the rank $n - \rho$. By suitable numeration, it becomes possible that exactly $\mathfrak{z}_1(t)\Big|_{t_1}^{t_1+P}, \ldots, \mathfrak{z}_{n-\rho}(t)\Big|_{t_1}^{t_1+P}$ are linearly independent. The remaining of these vectors can be linearly expressed by these vectors:

$$\delta_\mu(t)\Big|_{t_1}^{t_1+P} = \sum_{\sigma=1}^{n-\rho} c_{\mu,\sigma}\,\delta_\sigma(t)\Big|_{t_1}^{t_1+P} \quad \text{for} \quad \mu = n - \rho + 1, \ldots, n. \tag{30}$$

Consider now the vectors

$$\delta_m^*(t) = \begin{cases} \delta_m(t) & \text{for} \quad m = 1, 2, \ldots, n - \rho \\ \delta_m(t) - \sum_{\sigma=1}^{n-\rho} c_{m,\sigma}\,\delta_\sigma(t) & \text{for} \quad m = n - \rho + 1, \ldots, n. \end{cases} \tag{31}$$

Since <u>/12</u>

$$\mathrm{Det}\,(\delta_1^*(t), \ldots, \delta_n^*(t)) = \mathrm{Det}\,(\delta_1(t), \ldots, \delta_n(t)) \neq 0$$

it follows that the n vectors (31) are linearly independent solutions of eq.(4) for all t (see also theorem 4). The $\mathfrak{z}^*_\nu(t)$ with $\nu = 1, 2, \ldots, n - \rho$, because

of the linear independence of the vectors

$$\mathfrak{z}^{*}(t)\big|_{t}^{t+P}$$

including their linear combinations, are not periodic with P. Conversely, the $\mathfrak{z}_{\mu}^{*}(t)$ with $\mu = n - \rho + 1, \ldots, n$ have exactly the period P, as readily seen from eqs.(30), (4), and (2). This yields the proof for our case 2.

Case 3. Let $\rho = 0$. If eq.(4) had a solution periodic with P, also eq.(3) would have such a solution according to the above statements.

Section 4. The Resonance Case

Proof for Theorem 1. Thus, let $\mathfrak{z}(t)$ be a solution vector of eq.(4) periodic with P, for which eq.(5) is valid.

In contrast to the argument, we make the assumption:

$$|\mathfrak{x}(t)| \leq E \tag{32}$$

for all t and for an arbitrarily selected solution $\mathfrak{x}(t)$ of eq.(1). An integration of the first equation in the system (25) between t and t + mP, with arbitrary positive-whole m and taking eq.(5) into consideration, will yield

$$\mathfrak{z}^{T}(t)\,[\mathfrak{x}(t+mP) - \mathfrak{x}(t)] = m \cdot |\mathfrak{C}|. \tag{33}$$

Because of the periodicity of $\mathfrak{z}^{T}(t)$, a restriction of the following form exists for all t:

$$|\mathfrak{z}^{T}(t)| \leq D. \tag{34}$$

On the basis of this and from eq.(33), an estimate

$$D \cdot 2E \geq m|C|, \quad E \geq \frac{m|C|}{2D}.$$

is obtained. At sufficiently large m, this furnishes a contradiction to eq.(32).

The result can also be formulated as follows:

Theorem 9. In the resonance case, for each solution vector $\mathfrak{x}(t)$ of eq.(1)

and for each interval

$$t_0 \leq t \leq t_0 + m P \tag{35}$$

at least one point t^* exists for which

$$\xi(t^*) \geq \frac{m|C|}{2D} \tag{36}$$

vanishes with C from eq.(5) and D from eq.(34).

Section 5. The Principal Case

In this Section, it is assumed that eq.(4) has no solution $\vartheta(t)$ periodic with P. First, we will prove:

Theorem 10. If $\vartheta_1(t), \ldots, \vartheta_n(t)$ are linearly independent solutions of /13 eq.(4), we have

$$D(t) = \mathrm{Det}\begin{pmatrix} \delta_1^T(\tau)|_t^{t+P} \\ \delta_2^T(\tau)|_t^{t+P} \\ \vdots \\ \delta_n^T(\tau)|_t^{t+P} \end{pmatrix} \neq 0 \quad \textit{for all } t. \tag{37}$$

Proof. If, for a special quantity $t = t_2$, we would have $D(t_2) = 0$ in contrast to the argument, then n numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\alpha_1^2 + \ldots + \alpha_n^2 > 0$ could be so determined that

$$\sum_{\nu=1}^{n} \alpha_\nu \delta_\nu^T(t_2 + P) = \sum_{\nu=1}^{n} \alpha_\nu \delta_\nu^T(t_2)$$

Putting

$$\delta_0(t) = \sum_{\nu=1}^{n} \alpha_\nu \delta_\nu(t),$$

we would have

$$\delta_0(t_2 + P) = \delta_0(t_2).$$

However, according to eqs.(4) and (2), it would follow that

$$\delta_0(t + P) = \delta_0(t)$$

for all t, in contradiction to the assumption.

Proof of Theorem 2. If $\chi(t)$ were a solution of eq.(1) periodic with P, an integration of the first equation in the system (25) between a fixed point t_3 and $t_3 + P$ would yield

$$\mathfrak{y}_{\nu}^{T}(t)\Big|_{t_3}^{t_3+P}\mathfrak{x}(t_3)=\int_{t_3}^{t_3+P}\mathfrak{y}_{\nu}^{T}(t)\,\mathfrak{f}(t)\,dt \quad \text{for} \quad \nu=1,2,\ldots,n. \tag{38}$$

Since the coefficient determinant of this system of equations, according to theorem 10, differs from zero, the quantity $\mathfrak{x}(t_3)$ can be uniquely determined from this. Let us assume that eq.(1), with these initial values at the point t_3 is solved, thus yielding a solution vector $\mathfrak{x}(t)$. It only remains to be proved that this $\mathfrak{x}(t)$ is periodic with P. In view of eq.(2), it will then merely be necessary to determine that, automatically,

$$\mathfrak{x}(t_3+P)=\mathfrak{x}(t_3) \tag{39}$$

An integration of the first equation of the system (25) over t_3 and t_3 + P will yield, for ν = 1, 2, ..., n,

$$\mathfrak{y}_{\nu}^{T}(t_3+P)\cdot\mathfrak{x}(t_3+P)-\mathfrak{y}_{\nu}^{T}(t_3)\cdot\mathfrak{x}(t_3)=\int_{t_3}^{t_3+P}\mathfrak{y}_{\nu}^{T}(t)\,\mathfrak{f}(t)\,dt.$$

On deducting eq.(38) from this, the linear system of equations will read

$$\mathfrak{y}_{\nu}^{T}(t_3+P)\left[\mathfrak{x}(t_3+P)-\mathfrak{x}(t_3)\right]=0 \tag{40}$$

with a determinant differing from zero, analogous to eq.(8). This leads to vanishing of the bracket and thus of eq.(39).

Section 6. The Exceptional Case /14

If eq.(4) has exactly ρ linearly independent solutions $\mathfrak{z}_1(t)$, ..., $\mathfrak{z}_{\rho}(t)$ periodic with P, for which

$$\int_{t}^{t+P}\mathfrak{y}_{\nu}^{T}(t)\,\mathfrak{f}(t)\,dt=0, \qquad \nu=1,2,\ldots,\varrho \tag{41}$$

is valid, then the determinant (37) will exactly have the rank $n-\rho$. Of the system of equations (38), the first ρ equations (for ν = 1, 2, ..., ρ) are automatically satisfied since they contain only zeros. From the remaining

equations (38) with $\nu = \rho + 1, \ldots, n$, a total of ρ linearly independent vectors

$$\xi_1^*(t_0), \xi_2^*(t_0), \ldots, \xi_\rho^*(t_0) \tag{42}$$

can be determined, supplemented by the corresponding initial values (42) by solving eq.(1):

$$\xi_1^*(t), \xi_2^*(t), \ldots, \xi_\rho^*(t). \tag{43}$$

That, for each of these $\xi_\mu^*(t)$ ($\mu = 1, 2, \ldots, \rho$),

$$\xi_\mu^*(t_0 + P) = \xi_\mu^*(t_0) \tag{44}$$

is automatically obtained follows from the system of equations which is derived analogous to eq.(40):

$$\delta_\nu^T(t_0 + P)[\xi_\mu^*(t_0 + P) - \xi_\mu^*(t_0)] = 0 \tag{45}$$

with a determinant differing from zero. This means that all these $\xi_\mu^*(t)$ will have the period P. Thus, we can make the following statement:

Theorem 11. If eq.(4) has exactly ρ linearly independent solutions periodic with P, for each of which eq.(6) is valid, then eq.(1) has a ρ-parametric family of solutions periodic with P.

This result agrees with the trivial fact that all solution vectors of eq.(1) periodic with P are obtained by adding to one of these vectors all solution vectors of eq.(3) periodic with P.

PART II

THE RESONANCE CASE IN SYSTEMS OF n NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Section 1. Problem Formulation, Principal Results

As an extension of investigations made by R.Iglisch (Bibl.1), we consider here the nonlinear differential equation system

$$\frac{du_i}{dt} = g_i(u_1, u_2, \ldots, u_n; t) + h_i(t) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

which, under introduction of the vectors

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathfrak{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix} \quad \text{and} \quad g(u, t) = \begin{pmatrix} g_1(u_1, \ldots, u_n, t) \\ g_2(u_1, \ldots, u_n, t) \\ \vdots \\ g_n(u_1, \ldots, u_n, t) \end{pmatrix} \tag{2}$$

can be written in the following form:

$$\frac{du}{dt} = g(u, t) + \mathfrak{h}(t) \tag{3}$$

Let the periodicity assumption

$$g(u, t + P) = g(u, t), \quad \mathfrak{h}(t + P) = \mathfrak{h}(t). \tag{4}$$

be valid. For the functions g_i, in view of the variable u_i, let an expansion in Taylor series up to terms of the second order be possible, and let $\mathfrak{h}(t)$, for example, be continuous.

Assume that a solution $u_0(t)$ of eq.(3) periodic with P is known:

$$\frac{du_0}{dt} = g(u_0, t) + \mathfrak{h}(t), \quad u_0(t + P) = u_0(t). \tag{5}$$

Then, at a minor modification of $\mathfrak{h}(t)$, we will investigate the solution vectors $u(t)$ adjacent to $u_0(t)$ of the system of equations

$$\frac{du}{dt} = g(u, t) + \mathfrak{h}(t) + \beta \mathfrak{f}(t) \tag{6}$$

at sufficiently small $|\beta|$; here again, we assume

$$f(t+P)=f(t). \tag{7}$$

Using /18

$$u(t)=u_0(t)+\xi(t) \tag{8}$$

eq.(6) will be transformed, after subtraction of eq.(5), into

$$\frac{d\xi}{dt}=g(u_0(t)+\xi(t),t)-g(u_0(t),t)+\beta f(t). \tag{9}$$

Of these, the solutions with small $|\xi(t)|$ at $|\beta|$ will be investigated.

As will be demonstrated in Section 2 where a transformation of eq.(9) is to be made, the role of the homogeneous linear system of equations* is taken over by the system

$$\frac{d\eta}{dt}=\mathfrak{A}(t)\,\eta(t) \tag{10}$$

with the matrix (i, k = 1, 2, ..., n):

$$\mathfrak{A}(t)=(a_{ik}(t))=\left(\frac{\partial g_i(u_0,t)}{\partial u_k}\right)=\left(\frac{\partial g(u_0,t)}{\partial u_1},\ \frac{\partial g(u_0,t)}{\partial u_2},\ \ldots,\ \frac{\partial g(u_0,t)}{\partial u_n}\right). \tag{11}$$

Obviously,

$$\mathfrak{A}(t+P)=\mathfrak{A}(t). \tag{12}$$

The homogeneous system, adjoint to eq.(10), will then read as follows if the transition to the transposed matrix is denoted by the superscript T:

$$\frac{d\delta}{dt}=-\mathfrak{A}^T(t)\,\delta(t). \tag{13}$$

Theorem 1 (Resonance case). If eq.(13) has a solution vector $\xi(t)$ periodic with P, for which

$$\int_t^{t+P}\delta^T(t)\,f(t)\,dt=C\neq 0 \tag{14}$$

is obtained, each solution $\xi(t)$ of eq.(9) - independent of the initial values - will assume values with increasing t whose absolute amounts are at least of the order of magnitude of $\sqrt{|\beta|}$. The proof is given in Section 3.

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* See also another report by the same author (Bibl.2).

Then, we obtain the following in Section 4:

Theorem 2 (Principal case). If eq.(13) has no solution vector $\mathfrak{x}(t)$ periodic with P, then eq.(9) will have solution vectors $\xi(t)$ for each sufficiently small β which, for all values of t, retain the order of magnitude $|\beta|$, i.e., for which

$$|\xi(t)| \leq \text{Const.}\,|\beta| \tag{15}$$

is valid for all t, for example, the uniquely existing small solution periodic with P:

$$\xi(t+P) = \xi(t).$$

This only leaves the exceptional case to be defined which states that eq.(13) will have solution vectors $\mathfrak{x}_1(t)$, ..., $\mathfrak{x}_r(t)$ periodic with P at $1 \leq \leq r \leq n$ but that, for all these periodic solutions, the following expression is valid:

$$\int_t^{t+P} \mathfrak{x}_\varrho^T(t)\,\mathfrak{f}(t)\,dt = 0 \qquad (\varrho = 1, \ldots, r) \tag{16}$$

/19

In this case, it is impossible to make general statements on the functions $g_i(u, t)$ without further assumptions.

Section 2. A Transformation

Putting

$$u(t, \lambda) = u_0(t) + \lambda\,\xi(t), \tag{17}$$

will yield

$$u(t, 0) = u_0(t), \quad u(t, 1) = u_0(t) + \xi(t). \tag{18}$$

When applying the Taylor series with respect to λ to the difference on the right-hand side of the i^{th} component equation of the system (9), we obtain

$$g_i(u_0 + \xi, t) = g_i(u_0, t) + \frac{dg_i(u_0, t)}{d\lambda} + \int_0^1 \frac{d^2 g_i(u(t, \lambda), t)}{d\lambda^2} \cdot (1 - \lambda)\,d\lambda$$

or, if $x_k(t)$ are the components of the vector $\xi(t)$,

$$g_i(u_0+\xi,t)-g_i(u_0,t)=\sum_{k=1}^{n}\frac{\partial g_i(u_0,t)}{\partial u_k}x_k+ \\ +\int_0^1\sum_{k,l}^{1\ldots n}\frac{\partial^2 g_i(u(t,\lambda),t)}{\partial u_k\,\partial u_l}x_k x_l(1-\lambda)\,d\lambda. \tag{19}$$

In addition to the matrix $\mathfrak{A}(t)$ from eq.(11), we next introduce the "tensor" $T(u(t, \lambda), t)$ whose n components are to represent the matrices (k, ℓ = 1, ..., n):

$$\mathfrak{T}_i(u(t,\lambda),t)=\left(\frac{\partial^2 g_i(u(t,\lambda),t)}{\partial u_k\,\partial u_l}\right),\qquad i=1,\ldots,n \tag{20}$$

$$T(u(t,\lambda),t)=\begin{pmatrix}\mathfrak{T}_1(u(t,\lambda),t)\\ \mathfrak{T}_2(u(t,\lambda),t)\\ \vdots\\ \mathfrak{T}_n(u(t,\lambda),t)\end{pmatrix} \tag{21}$$

Then, eq.(9) can be written in the following form:

$$\frac{d\xi(t)}{dt}=\mathfrak{A}(t)\xi(t)+\beta f(t)+\int_0^1\xi^T(t)\,T(u(t,\lambda),t)\,\xi(t)\,(1-\lambda)\,d\lambda \tag{22}$$

Of these, at sufficiently small ϵ and β_0, solutions will be sought with

$$|\xi(t)|\leq\epsilon\quad\text{at}\quad|\beta|\leq\beta_0. \tag{23}$$

It should be noted here that the tensor

$$T(u(t,0),t)=T(u_0(t),t),$$

has the period P:

$$T(u_0(t+P),t+P)=T(u_0(t),t). \tag{24}$$

From this fact, in combination with the first relation (23), the existence of a finite constant M follows in such a manner that, for the integral in eq.(22), the following is valid: /20

$$\left|\int_0^1\xi^T(t)\,T(u(t,\lambda),t)\,\xi(t)\,(1-\lambda)\,d\lambda\right|\leq M\cdot(\max|\xi(t)|)^2 \tag{25}$$

For this, it merely must be assumed that all second derivatives on the right-hand side in eq.(20), within an interval of $t_1 \leq t \leq t_1 + P$, are restricted for all values $|u(t, \lambda) - u_0(t)| \leq \epsilon$ of the first argument.

Section 3. The Resonance Case

Let the adjoint system (13) have a solution vector $\mathfrak{z}(t)$ periodic with P, for which eq.(14) is valid. In contrast to the argument of theorem 1, we are making the following assumption:

$$|\mathfrak{x}(t)| \leq N \sqrt{|\beta|} \tag{26}$$

for an arbitrary solution $\mathfrak{x}(t)$ which satisfies the first stipulation (23) for all t with a finite constant N, to be determined later. Analogous to eq.(33) in another paper (Bibl.2), eqs.(22) and (13), for an arbitrary positive-whole n with C, will yield on the basis of eq.(14):

$$\mathfrak{z}^T(t)\,[\mathfrak{x}(t+mP)-\mathfrak{x}(t)] = mC\cdot\beta + \\ + \int_t^{t+mP} \mathfrak{z}^T(\tau) \int_0^1 \mathfrak{x}^T(\tau)\, T(u(\tau,\lambda),\tau)\, \mathfrak{x}(\tau)\,(1-\lambda)\, d\lambda\, d\tau. \tag{27}$$

Because of the periodicity of $\mathfrak{z}^T(t)$, a constraint of the following form applies to all t:

$$|\mathfrak{z}^T(t)| \leq D. \tag{28}$$

If eqs.(28), (26), and (25) are used, an estimate according to eq.(27) will be

$$D\cdot 2N|\beta|^{\frac{1}{2}} + DMN^2|\beta|\,mP \geq m|\beta||C|$$

or

$$2DN \geq m|\beta|^{\frac{1}{2}}\cdot B \tag{29}$$

with

$$B = |C| - DMN^2P > 0. \tag{30}$$

If a sufficiently small constant is substituted for N, then B will be positive. For each $\beta \neq 0$, the quantity m can be selected so large that eq.(29) will contain a contradiction. This proves that the assumption (26), in combination with the first relation (23), is impossible and that, therefore, the following theorem applies:

Theorem 3. In the resonance case, four constants ϵ, M, D, N exist, from which because of eq.(30) a positive B can be determined in such a manner that

each solution $\xi(t)$ of eq.(9), within each interval of the length $\frac{2 \cdot DN}{B|\beta|^{\delta}} P$, assumes at least once a value so that

$$|\xi(t)| > \operatorname{Min}(\varepsilon, N\sqrt[\delta]{|\beta|}) \tag{31}$$

is justified.

Lemma. If β is restricted by /21

$$|\beta| < \frac{\varepsilon^{\delta}}{N^{\delta}}, \text{ with } N^{\delta} < \frac{|C|}{DMP} \quad [\text{see eq.(30)}]$$

then, in each interval of the length $\frac{2DN}{B|\beta|^{\delta}} P$, the following estimate will apply for at least one value of t:

$$|\xi(t)| > N\sqrt[\delta]{|\beta|}. \tag{32}$$

Section 4. The Principal Case

In the following, we will require the main theorem on implicit functions as an auxiliary theorem.

Auxiliary theorem 1. Let the vector

$$\mathfrak{v}(a, \beta) = \mathfrak{v}(a_1, a_2, \ldots, a_n, \beta) = \begin{pmatrix} v_1(a_1, \ldots, a_n, \beta) \\ v_2(a_1, \ldots, a_n, \beta) \\ \vdots \\ v_n(a_1, \ldots, a_n, \beta) \end{pmatrix} \tag{33}$$

in a certain vicinity of the quantities

$$a_1 = 0, \ a_2 = 0, \ \ldots, \ a_n = 0, \ \beta = 0$$

possess continuous first derivatives to a_ν and let this same vector be continuous in all n + 1 variables. In addition, let

$$\mathfrak{v}(0, \ldots, 0, 0) = \mathfrak{o}; \tag{34}$$

finally be the functional determinant

$$\left| \frac{\partial v_i(0, \ldots, 0, 0)}{\partial a_k} \right| \neq 0 \qquad (i, k = 1, 2, \ldots, n). \tag{35}$$

Thus, for each sufficiently small a_0 a constraint β_0 will exist such that, for

each β with $|\beta| \leq \beta_0$, a solution vector $\mathfrak{a} = \mathfrak{a}(\beta)$ of the systems of equations

$$\mathfrak{v}(\mathfrak{a}, \beta) = \mathfrak{o} \tag{36}$$

exists uniquely, for which $|\mathfrak{a}| \leq a_0$. For the proof, see for example another paper (Bibl.3).

This auxiliary theorem is applied as follows: Each solution vector $\mathfrak{x}(t)$ of eq.(9) is characterized by its initial values:

$$\mathfrak{x}(0) = \mathfrak{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \tag{37}$$

i.e.,

$$\mathfrak{x} = \mathfrak{x}(\mathfrak{a}, \beta; t). \tag{38}$$

With respect to t, this vector has the period P if and only if

$$\mathfrak{v}(\mathfrak{a}, \beta) = \mathfrak{x}(\mathfrak{a}, \beta; P) - \mathfrak{x}(\mathfrak{a}, \beta; 0) = \mathfrak{o} \tag{39}$$

This corresponds to the argument (36) of the auxiliary theorem. For $\beta = 0$, /22 the null vector represents the solution of eq.(9); consequently, eq.(34) is satisfied since then also $\mathfrak{a}$ is the null vector. The assumptions as to derivation and continuity of the auxiliary theorem are ensured by the following auxiliary theorem, in accordance with known theorems as to the dependence of the solution vectors of eq.(9) on the initial values $\mathfrak{a}$ and on the parameter β (Bibl.4).

Auxiliary theorem 2. Within the interval $0 \leq t \leq P$, for a given ϵ, one positive ϵ_1 and β_1 each can be determined such that the solutions $\mathfrak{x}(t)$ of eq.(9) satisfy the estimate $|\mathfrak{x}(t)| \leq \epsilon$ in the entire interval if only $|\mathfrak{x}(0)| \leq \epsilon_1$ and $|\beta| \leq \beta_1$ are selected.

In accordance with eq.(39), it is necessary, for application of the auxiliary theorem 1, that $\epsilon_1 \leq a_0$ as well as $\beta_1 \leq \beta_0$. A differentiation of eq.(9) to

a_l indicates that the vectors

$$\eta_i^{*}(t) = \frac{\partial \xi(a, \beta; t)}{\partial a_l} \qquad (l = 1, 2, \ldots, n) \tag{40}$$

are solutions of the system of equations

$$\frac{d\eta^{*}}{dt} = \mathfrak{B}(t)\,\eta^{*} \tag{41}$$

with the matrix

$$\mathfrak{B}(t)_{ik} = \left(\frac{\partial g_i(u_0(t) + \xi(t), t)}{\partial u_k}\right) \qquad (i, k = 1, \ldots, n). \tag{42}$$

Therefore, the vectors

$$\eta_i^{0}(t) = \frac{\partial \xi(0, \ldots, 0, 0; t)}{\partial a_l} \qquad (l = 1, \ldots, n) \tag{43}$$

are solutions of eq.(10) since, in that case, eq.(42) passes over into eq.(11). The vectors (43) are linearly independent since their determinant for t = 0 is the unit determinant [see also theorem 4 in another paper (Bibl.2)]. Since, in the principal case, we make the assumption that eq.(13) and thus also eq.(10) [see theorem 8 (Bibl.2)] has no solution vectors periodic with P, it follows that [see theorem 10 (Bibl.2)]

$$\operatorname{Det}\begin{pmatrix} \eta_1^{0}(\tau)\big|_0^{\tau+P} \\ \eta_2^{0}(\tau)\big|_0^{\tau+P} \\ \vdots \\ \eta_n^{0}(\tau)\big|_0^{\tau+P} \end{pmatrix} \neq 0. \tag{44}$$

A brief examination of eq.(39) indicates that this is exactly the condition (35) of the auxiliary theorem. Since all conditions of the auxiliary theorem are proved to be valid, its argument - in our case, eq.(39) - can be considered as also proved.

Therefore, the following applies:

Theorem 4. If eq.(13) has no solution vector $\mathfrak{x}(t)$ periodic with P, constraints ϵ_1 and β_1 will exist for a given ϵ such that an initial vector $\xi(0)$ with $|\xi(0)| \leq \epsilon_1$ exists for each β with $|\beta| \leq \beta_1$, so that the solution $\xi(t)$ is periodic with this initial value $\xi(0)$ and satisfies the uniform estimate

$|\xi(t)| \leq \epsilon$.

This represents a portion of the theorem 2. To prove this theorem completely, we will need still another theorem. /23

Theorem 5. Assume that a finite constant E exists such that, at sufficiently small $|\beta|$, the following will be valid for this solution $\xi(t)$ of eq.(9) periodic with P:

$$|\xi(t)| \leq E \cdot |\beta|. \tag{45}$$

Proof. If $\eta_1(t)$, ..., $\eta_n(t)$ are linearly independent solutions of eq.(13), the following expression is obtained under utilization of the periodicity of $\xi(t)$ analogous to eq.(38) (Bibl.2) for $\nu = 1, \ldots, n$:

$$\begin{aligned}[\delta_\nu^T(t+P) - \delta_\nu^T(t)]\,\xi(t) = \beta \int_t^{t+P} \delta_\nu^T(\tau)\, f(\tau)\, d\tau + \\ + \int_t^{t+P} \delta_\nu^T(\tau) \int_0^1 \xi^T(\tau)\, T(u(\tau,\lambda),\tau)\, \xi(\tau)\,(1-\lambda)\, d\lambda\, d\tau\end{aligned} \tag{46}$$

[see also eq.(27)]. The determinant of the coefficient matrix on the left-hand side, according to the assumption of the principal case, differs from zero [see eq.(44)]. Therefore, the linear system of equations (46) can be solved for the components $x_1(t)$, ..., $x_n(t)$ of $\xi(t)$ on the left-hand side, using Cramer's solution formula. If, at fixed t,

$$x = \operatorname*{Max}_{t \leq \tau \leq t+P} |x_\nu(\tau)| \qquad (\nu = 1, 2, \ldots, n), \tag{47}$$

then an estimate of the following form will be obtained in this manner:

$$x \leq K|\beta| + L \cdot (\max|\xi(t)|)^2 \tag{48}$$

with two finite constants K and L, taking eq.(25) into consideration.

Since

$$\max|\xi(t)| \leq \sqrt{n}\, x$$

it follows from eq.(48) even more so that

i.e.,

$$\max|\xi(t)| \leq \sqrt{n}\, K|\beta| + \sqrt{n}\, L \cdot (\max|\xi(t)|)^2,$$

$$\max|\xi(t)|\,[1 - \sqrt{n}\, L \cdot (\max|\xi(t)|)] \leq \sqrt{n}\, K|\beta|.$$

On the basis of theorem 4, we can then assume that the expression in brackets is positive. For example, for $\varepsilon \leq \frac{1}{2\sqrt{n}L}$, this expression becomes $\geq \frac{1}{2}$. From this, the estimate (45) with $E = \sqrt{n}K$ is immediately obtained.

Instead of using Cramer's rule for the solution, it is naturally also possible to solve eq.(46) for $\xi(t)$ by left-hand multiplication with the matrix inverse to $[\mathfrak{z}_\nu^T(t + P) - \mathfrak{z}_\nu^T(t)]$ and then to make the estimate by means of Schwarz' inequality $|\xi(t)|$. This will also yield eq.(45).

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PART III

/27

STUDY OF THE RESONANCE CASE IN SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Section 1. Introduction

Previously (Bibl.1) we investigated the linear inhomogeneous differential equation system

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{A}(t)\mathfrak{x} + \mathfrak{f}(t) \tag{1}$$

with a n-row matrix $\mathfrak{A}(t)$ which, for simplicity, was assumed as being continuous and having a continuous n-component vector $\mathfrak{f}(t)$ of the same period P

$$\mathfrak{A}(t + P) = \mathfrak{A}(t), \quad \mathfrak{f}(t + P) = \mathfrak{f}(t) \tag{2}$$

The corresponding homogeneous system then will be

$$\frac{d\mathfrak{y}}{dt} = \mathfrak{A}(t)\mathfrak{y} \tag{3}$$

and the adjoint system

$$\frac{d\mathfrak{z}}{dt} = -\mathfrak{A}^{T}(t)\mathfrak{z}, \tag{4}$$

where the superscript T denotes the transition to the transposed matrix.

In all, three cases were differentiated: The principal case is present if eq.(4) has no solution periodic with P; the resonance case is present if eq.(4) has at least one solution vector $\mathfrak{z}(t)$ periodic with P, for which

$$\int_0^P \mathfrak{z}^{T}(\tau)\,\mathfrak{f}(\tau)\,d\tau = c \neq 0 \tag{5}$$

is valid; the exceptional case is present if eq.(4) does have periodic solutions $\mathfrak{z}_1(t), \mathfrak{z}_2(t), \ldots, \mathfrak{z}_\rho(t)$ $(1 \leq \rho \leq n)$ periodic with P but if the following is valid for all these $\mathfrak{z}_\mu(t)$:

$$\int_0^P \mathfrak{z}_\mu^{T}(\tau)\,\mathfrak{f}(\tau)\,d\tau = 0 \quad \text{for } \mu = 1,2,\ldots,\rho. \tag{6}$$

Whereas, in the principal case as well as in the exceptional case, solutions

of eq.(1) periodic with P, i.e., remaining limited for all values of t, are in existence, the values of all solution vectors $\mathfrak{x}(t)$ of eq.(1) tend toward infinite with increasing t in the resonance case, independent of the initial values. The individual steps in this process will be further investigated in the present paper.

We use the following notations /28

$$\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_n), \text{ resp. } \mathfrak{Z} = (\mathfrak{y}^{-1})^T \tag{7}$$

for denoting a fundamental system of solutions of eqs.(3) and (4), respectively [see eq.(17) in a previous report (Bibl.1)]. In the method of variation of the constants*, using the abbreviation

$$\mathfrak{z}^*_\nu(t) = \mathfrak{y}_\nu(t) \cdot c_\nu(t) \quad (\nu = 1,2,\ldots,n) \tag{8}$$

the following argument is constructed for the solution of eq.(1):

$$\mathfrak{z}(t) = \sum_{\nu=1}^{n} \mathfrak{z}^*_\nu(t) = \mathfrak{y}(t) \cdot \mathfrak{c}(t) \tag{9}$$

In the present report, an argument similar to eq.(8), namely,

$$\mathfrak{z}(t) = \sum_{\nu=1}^{s} \mathfrak{z}^{(\nu)}_*(t), \quad s \leqq n$$

[see eqs.(77) and (78)] is discussed, where the $\mathfrak{x}^{(\nu)}_*(t)$ are composed in a suitable manner from the $\mathfrak{x}^*_\nu(t)$. In the resonance case, statements can be made on the behavior of these vectors $\mathfrak{x}^{(\nu)}_*(t)$ for t increasing without bounds. Here, the matrix

$$\mathfrak{F} = \mathfrak{y}^{-1}(t) \cdot \mathfrak{y}(t + P), \tag{10}$$

which had been discussed in Section 2 plays a decisive role. By a suitable transformation, the system (1) can be brought to a "normal form" (Section 3) with constant coefficients, by means of which the study of the vectors $\mathfrak{x}^{(\nu)}_*(t)$

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* With respect to the method of variation of the constants, see footnote on P. 58.

can be made much more concise (Section 4). Making use of this method, four numerical examples will be given at the end of this paper (Section 5).

A brief report, to be published soon, will apply the above considerations to the case of an ordinary differential equation of the n^{th} order with periodic coefficients. In that paper, additional examples will be given.

Section 2. The Matrix $\mathfrak{B}$ /29

Theorem 1: The matrix $\mathfrak{B}$, defined in eq.(10), is a constant regular matrix, i.e., a matrix independent of t, under the first assumption [eq.(2)].

Proof: In view of eq.(7), eq.(3) can be written in the form of

$$\mathfrak{y}' = \mathfrak{A}(t)\mathfrak{y} \tag{11}$$

Then, a fundamental solution matrix of eq.(4) will be as follows [see (Bibl.1), eq.(17)]:

$$\mathfrak{Z} = (\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_n) = (\mathfrak{y}^{-1})^T, \tag{12}$$

which means that

$$\mathfrak{Z}' = -\mathfrak{A}^T(t)\mathfrak{Z}. \tag{13}$$

is valid. According to eqs.(10), (12), (13), and (11) we then obtain

$$\mathfrak{P}' = \left(\mathfrak{Z}^T(t)\,\mathfrak{y}(t+P)\right)' = \mathfrak{Z}'^T(t)\mathfrak{y}(t+P) + \mathfrak{Z}^T(t)\,\mathfrak{y}'(t+P) =$$
$$= -\mathfrak{Z}^T(t)\,\mathfrak{A}(t)\,\mathfrak{y}(t+P) + \mathfrak{Z}^T(t)\cdot\mathfrak{A}(t+P)\mathfrak{y}(t+P) = \mathfrak{O},$$

when taking the first relation (2) into consideration. Consequently, $\mathfrak{B}$ is a constant matrix. That its determinant differs from zero follows from the non-vanishing of the two determinants of the matrices on the right-hand side of eq.(10).

Theorem 2: If we pass from a fundamental solution matrix $\mathfrak{y}(t)$ of eq.(11), on multiplying on the right-hand side by a regular constant matrix $\mathfrak{C}$, to a new fundamental system

$$\mathfrak{Y}^{\circ}(t) = \mathfrak{Y}(t).\mathfrak{L} \tag{14}$$

[see (Bibl.1), Theorem 5], the following relation will be valid for the matrix $\mathfrak{P}^{\circ}$ which had been formed in accordance with eq.(10):

$$\mathfrak{P}^{\circ} = \mathfrak{L}^{-1}\mathfrak{P}\mathfrak{L} . \tag{15}$$

Proof: From eq.(10) and (14), we obtain

$$\mathfrak{P}^{\circ} = \mathfrak{Y}^{\circ -1}(t)\,\mathfrak{Y}^{\circ}(t+P) = \mathfrak{L}^{-1}\mathfrak{Y}^{-1}(t)\,\mathfrak{Y}(t+P)\mathfrak{L} = \mathfrak{L}^{-1}\mathfrak{P}\mathfrak{L}.$$

This means that, by a suitable selection of the fundamental system $\mathfrak{Y}(t)$, the matrix $\mathfrak{P}$ can be transformed into any normal form which can be produced by a similarity transformation of the type of eq.(15).

Theorem 3: Let the constant matrix $\mathfrak{P}$ be given in the Jordan normal form /30 [see (Bibl.2), Sect.19.1]

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{P}_1 & & & \\ & \mathfrak{P}_2 & & \\ & & \ddots & \\ & & & \mathfrak{P}_s \end{pmatrix} \quad \text{with} \quad \mathfrak{P}_\nu = \begin{pmatrix} \lambda_\nu & 1 & & \\ & \lambda_\nu & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & \lambda_\nu \end{pmatrix} \tag{16}$$

where one elementary component $\mathfrak{P}_\nu$ of the order $m_\nu \geq 1$ in the main diagonal has the eigenvalue λ_ν and contains ones in the next higher diagonal (zeros are not entered). It is obvious that

$$n = m_1 + m_2 + \ldots + m_s . \tag{17}$$

Then, it is possible to form a matrix $\mathfrak{N}_q^*$ with any real number q, so that

$$\mathfrak{P} = e^{\mathfrak{N}_q^* \cdot q} \tag{18}$$

is valid and that $\mathfrak{N}_q^*$ has a normal form of

$$\mathfrak{A}_q^* = \begin{pmatrix} \mathfrak{A}_{q1}^* & & & \\ & \mathfrak{A}_{q2}^* & & \\ & & \ddots & \\ & & & \mathfrak{A}_{qs}^* \end{pmatrix} \text{ with } \mathfrak{A}_{q\nu}^* = \begin{pmatrix} \alpha_\nu & \beta_\nu^{(1)} & \beta_\nu^{(2)} & \cdots & \beta_\nu^{(m_\nu-1)} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \beta_\nu^{(2)} \\ & & & \ddots & \beta_\nu^{(1)} \\ & & & & \alpha_\nu \end{pmatrix} \tag{19}$$

where

$$\alpha_\nu = \frac{1}{q} \ln \lambda_\nu \text{ with } -\frac{\pi}{q} < \mathfrak{J}(\alpha_\nu) \leqq \frac{\pi}{q} \tag{20}$$

and

$$\beta_\nu^{(\sigma)} = (-1)^{\sigma-1} \frac{1}{\sigma \cdot q \cdot \lambda_\nu^\sigma} \text{ for } \sigma = 1,2,\ldots m_\nu - 1. \tag{21}$$

Proof: Let us form [see also, for example, (Bibl.3), pp.333/4] /31

$$\ln \mathfrak{A}_\nu = \ln(\lambda_\nu \mathfrak{E}_\nu + \mathfrak{Q}_\nu), \tag{22}$$

where $\mathfrak{E}_\nu$ is the unit matrix of rank m_ν while the m_ν-row matrix

$$\mathfrak{Q}_\nu = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \tag{23}$$

has the rank $m_\nu - 1$. By expansion in series, we obtain*

$$\ln \mathfrak{A}_\nu = \ln(\lambda_\nu \mathfrak{E}_\nu) + \ln(\mathfrak{E}_\nu + \frac{1}{\lambda_\nu}\mathfrak{Q}_\nu) =$$

$$= (\ln \lambda_\nu)\cdot \mathfrak{E}_\nu + \ln(\mathfrak{E}_\nu + \frac{1}{\lambda_\nu}\mathfrak{Q}_\nu)$$

$$= (\ln \lambda_\nu)\cdot \mathfrak{E}_\nu + \frac{\mathfrak{Q}_\nu}{\lambda_\nu} - \frac{\mathfrak{Q}_\nu^2}{2\lambda_\nu^2} + \frac{\mathfrak{Q}_\nu^3}{3\lambda_\nu^3} \mp \cdots$$

$$\begin{pmatrix} \ln \lambda_\nu & \frac{1}{\lambda_\nu} & -\frac{1}{2\lambda_\nu^2} & \cdots & (-1)^{m_\nu-2}\frac{1}{(m_\nu-1)\lambda_\nu^{m_\nu-1}} \end{pmatrix}$$

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* The formula

$$\ln(\lambda_\nu \mathfrak{E}_\nu) = (\ln \lambda_\nu)\cdot \mathfrak{E}_\nu$$

used here is equivalent to

$$\lambda_\nu \mathfrak{E}_\nu = e^{(\ln \lambda_\nu)\cdot \mathfrak{E}_\nu} = e^{\ln \lambda_\nu} \cdot \mathfrak{E}_\nu = \lambda_\nu \mathfrak{E}_\nu,$$

as is readily verified by means of a power series. On the basis of this fact, we can also write in abbreviated form:

$$e^{\alpha}\cdot \mathfrak{E}_\nu = e^{\alpha \mathfrak{E}_\nu}.$$

$$= \begin{pmatrix} \ln\lambda_\nu & \frac{1}{\lambda_\nu} & \cdots & (-1)^{\nu-3}\frac{1}{(\nu-2)\lambda_\nu^{\nu-2}} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \frac{1}{\lambda_\nu} \\ & & & \ln\lambda_\nu \end{pmatrix}$$

[see also, for example (Bibl.4), p.82]. For this, it is suggested to use the /32 principal value of the natural logarithm ln, i.e.,

$$-\pi < \mathcal{J}(\ln\lambda_\nu) \leq \pi \,, \tag{24}$$

which agrees with eq.(20). Taking eqs.(21) and (19) into consideration, we will obtain

$$\frac{1}{q}\ln\psi_\nu = \mathfrak{K}_{q\nu}^{*} \,, \tag{25}$$

from which, by means of expanding the exponential function in a series [with respect to the convergence, see for example (Bibl.4), p.119],

$$\psi_\nu = e^{\mathfrak{K}_{q\nu}^{*}\cdot q} \tag{26}$$

is obtained. From this, eq.(18) with $\mathfrak{K}_q^*$ can then be taken from eq.(19) since, in forming the powers of $\mathfrak{K}_q^*$, the elementary components $\mathfrak{K}_{q\nu}^*$ do not exert a mutual influence.

By an additional collineatory transformation, the matrix $\mathfrak{K}_q^*$ in eq.(19) can be brought to the Jordan normal form. If $\mathfrak{C}$ represents the matrix of this similarity transformation, i.e.,

$$\mathfrak{K}_q = \mathfrak{C}^{-1}\,\mathfrak{K}_q^*\,\mathfrak{C} \,,$$

then $\mathfrak{K}$ is transformed into $\mathfrak{C}^{-1}\mathfrak{K}\mathfrak{C}$, as readily demonstrated by expanding the exponential function in a series. We can now write

$$\mathfrak{A}_q = \begin{pmatrix} \mathfrak{A}_{q1} & & & \\ & \mathfrak{A}_{q2} & & \\ & & \ddots & \\ & & & \mathfrak{A}_{qs} \end{pmatrix} \text{ with } \mathfrak{A}_{q\nu} = \begin{pmatrix} \alpha_\nu & 1 & & \\ & \alpha_\nu & 1 & \\ & & \ddots & \ddots \\ & & & \alpha_\nu \end{pmatrix}. \tag{27}$$

This will yield the following:

Theorem 4: The matrix $\mathfrak{A}$ can be brought to the following form by a similarity transformation: /33

$$\mathfrak{A} = e^{\mathfrak{A}_q \cdot q} \tag{28}$$

with $\mathfrak{A}_q$ from eq.(27).

Lemma: If, specifically, all elementary components of the matrix $\mathfrak{A}$ have the order 1, also $\mathfrak{A}_q$ will be a diagonal matrix.

Definition: Below, we will replace the arbitrary number q by the period P of the coefficients of eq.(1); in this case, we can write in abbreviated form:

$$\mathfrak{A}_p = \mathfrak{A}, \mathfrak{A}_p^* = \mathfrak{A}^*, \mathfrak{A} = e^{\mathfrak{A} P}. \tag{29}$$

Theorem 5: By means of the constant matrix $\mathfrak{A}$ [see eq.(29) and (27)], the fundamental system $\mathfrak{Y}(t)$ of eq.(3) can be written in the form

$$\mathfrak{Y}(t) = \Phi(t) \cdot e^{\mathfrak{A} t} \text{ with } \Phi(t) = \left(\bar{\varphi}_1(t), \bar{\varphi}_2(t), \ldots, \bar{\varphi}_n(t)\right) \tag{30}$$

where $\Phi(t)$ has the period P. Analogously, the fundamental system $\mathfrak{Z}(t)$ of eq.(4) can be brought to the form

$$\mathfrak{Z}(t) = \mathfrak{Y}(t) \left(e^{-\mathfrak{A} \cdot t}\right)^T \tag{31}$$

where

$$\mathfrak{Y}(t) = \left(\Phi^{-1}\right)^T = \left(\bar{\psi}_1(t), \bar{\psi}_2(t), \ldots, \bar{\psi}_n(t)\right) \tag{32}$$

also has the period P.

Proof: It is to be demonstrated that, in the argument (30),

$$\Phi(t) = \mathfrak{Y}(t) \cdot e^{-\mathfrak{A} t} = \left(\bar{\varphi}_1(t), \bar{\varphi}_2(t), \ldots, \bar{\varphi}_n(t)\right) \tag{33}$$

has the period P. Making use of eqs.(10), (29), and (30), we obtain

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$$\Phi(t+P) = \Psi(t+P)e^{-\alpha P}\cdot e^{-\alpha t}\cdot\Psi(t)\,\Psi^{-1}\cdot e^{-\alpha P}\cdot e^{-\alpha t} = \Phi(t).$$

Then, the relations (31) and (32) are directly obtained from eqs.(12) and (30).

The conventional way of constructing the constant matrix $\mathfrak{A}$, described in developing the theorem 3, usually is quite time-consuming. That this procedure can be less cumbersome in certain special cases is indicated by the following theorem*.

Theorem 6: If the matrix $\mathfrak{A}(t)$ with its integral from 0 to t, periodic with P, is transposable, i.e., if the relation

$$\mathfrak{A}(t)\cdot\int_0^t\mathfrak{A}(\tau)d\tau = \int_0^t\mathfrak{A}(\tau)d\tau\cdot\mathfrak{A}(t) \tag{34}$$

exists identically in t, then the constant matrix

$$\mathfrak{K} = \frac{1}{P}\int_0^P\mathfrak{A}(\tau)d\tau \tag{35}$$

will yield

$$\Psi = e^{\mathfrak{K}^{*}t}. \tag{36}$$

To this belongs the fundamental solution matrix

$$\Psi(t) = e^{\int^t\mathfrak{A}(\tau)d\tau} = \Phi^{*}\cdot e^{\mathfrak{K}^{*}t} \tag{37}$$

with the following matrix, periodic with P,

$$\Phi^{*}(t) = e^{\int^t(\mathfrak{A}(\tau)-\mathfrak{K}^{*})d\tau}. \tag{38}$$

Proof: Below, the two formulas which can be proved by expanding the exponential function in a series are used [see also, for example (Bibl.4), /35

* I wish to express my thanks to Dr.H.Eltermann for developing the idea of theorems 6, 7, and 8.

pp.121 and 138]:

$$e^{B+C} = e^{B} \cdot e^{C}, \quad \text{if } BC = CB; \tag{39}$$

$$\left(e^{B(t)}\right)' = e^{B(t)} \cdot B'(t) = B'(t) \cdot e^{B(t)}, \quad \text{if } B'B = BB'. \tag{40}$$

First, it can be demonstrated on the basis of eq.(40) that $\mathfrak{N}(t)$, according to eq.(37), is a fundamental solution matrix of eq.(11). [But note: $\mathfrak{N}(0) = \mathfrak{E}$.] The constant matrix $\mathfrak{B}$ (see theorem 1) can then be calculated from eq.(10) with $t = 0$, yielding

$$\mathfrak{B} = \ln \mathfrak{Y}(P) = e^{\int_0^P \mathfrak{A}(\tau)\,d\tau}, \tag{41}$$

from which eq.(36) is obtained with eq.(35).

It only remains to prove that the matrix $\Phi^{**}(t)$ defined by the argument (37) in accordance with eq.(30), can be written in the form of eq.(38). Because of eq.(35), $\Phi^{**}(t)$ has the period P, since

$$\int_0^P \left(\mathfrak{A}(\tau) - \mathfrak{K}^{**}\right) d\tau = \mathfrak{O} \tag{42}$$

The relations (37) and (38) can be reduced to the identity

$$e^{\int_0^t \mathfrak{A}(\tau)\,d\tau} = e^{\int_0^t \left(\mathfrak{A}(\tau) - \mathfrak{K}^{**}\right) d\tau} \cdot e^{\mathfrak{K}^{**} t}, \tag{43}$$

if it can be demonstrated [see eq.(39)] that the matrix $\mathfrak{K}^{**}$ is commutative with the matrix $\int_0^t \mathfrak{A}(\tau)\,d\tau$, i.e., if

$$\int_0^P \mathfrak{A}(\tau)\,d\tau \cdot \int_0^t \mathfrak{A}(\sigma)\,d\sigma = \int_0^t \mathfrak{A}(\sigma)\,d\sigma \cdot \int_0^P \mathfrak{A}(\tau)\,d\tau \tag{44}$$

is valid. Since this equation is directly understandable for $t = 0$, eq.(44) will be obtained by differentiation to t, based on the commutability relation

$$\int_0^P \mathfrak{A}(\tau)\,d\tau \cdot \mathfrak{A}(t) = \mathfrak{A}(t) \cdot \int_0^P \mathfrak{A}(\tau)\,d\tau. \tag{45}$$

Taking the periodicity of $\mathfrak{A}(t)$ into consideration, we can calculate /36

$$\mathfrak{A}(t)\int_0^P \mathfrak{A}(\tau)\,d\tau = \mathfrak{A}(t)\left(\int_0^{t+P}\mathfrak{A}(\tau)\,d\tau - \int_P^{t+P}\mathfrak{A}(\tau)\,d\tau\right) = \mathfrak{A}(t+P)\int_0^{t+P}\mathfrak{A}(\tau)\,d\tau - \mathfrak{A}(t)\cdot\int_0^t \mathfrak{A}(\tau)\,d\tau \tag{46}$$

and, taking eq.(34) into consideration,

$$= \int_0^{t+P}\mathfrak{A}(\tau)\,d\tau\;\mathfrak{A}(t+P) - \int_0^t \mathfrak{A}(\tau)\,d\tau\cdot\mathfrak{A}(t) = \left(\int_0^{t+P}\mathfrak{A}(\tau)\,d\tau - \int_P^{t+P}\mathfrak{A}(\tau)\,d\tau\right)\mathfrak{A}(t) = \int_0^P \mathfrak{A}(\tau)\,d\tau\cdot\mathfrak{A}(t),$$

which proves eqs.(44) and thus also eq.(43).

By a similarity transformation, the matrix (35) can now be brought to the Jordan normal form, indicated in eq.(27), in such a manner that the sequence of $\alpha_1, \alpha_2, \ldots, \alpha_s$ is the same. This will yield a matrix $\mathfrak{K}^\circ$, for which

$$e^{\mathfrak{K}^\circ P} = \mathfrak{F} = e^{\mathfrak{K}P} \tag{47}$$

with $\mathfrak{K}$ from eq.(29) is valid. The eigenvalues of the matrices (47) thus are mutually equal. Accordingly, the eigenvalues α_ν of $\mathfrak{K}$ and $\mathfrak{K}^\circ$ can differ at most by $n_\nu \cdot \frac{2\pi_i}{P}$ with integral n_ν. Thus, we have

$$\mathfrak{K}^\circ = \mathfrak{K} + \begin{pmatrix} \mathfrak{k}_{\vartheta_1} & & \\ & \mathfrak{k}_{\vartheta_2} & \\ & & \ddots \\ & & & \mathfrak{k}_{\vartheta_s} \end{pmatrix} \quad \text{with} \quad \mathfrak{k}_{\vartheta_\nu} = n_\nu \cdot \frac{2\pi i}{P}\,\mathfrak{k}_\nu \tag{48}$$

[see also eq.(27)], if $\mathfrak{E}_\nu$ denotes the m_ν-row unit matrix [see eq.(17)]. This will lead to the following theorem:

Theorem 7: By a similarity transformation, $\mathfrak{K}^{**}$ from eq.(35) can be /37 transformed into a matrix $\mathfrak{K}^\circ$ which is correlated with eq.(48) over $\mathfrak{K}$ from eq.(29), so that $\mathfrak{K}$ can be calculated.

Theorem 8: The assumption (34) for the theorems 6 and 7 can be replaced by the stronger assumption

$$A(t)\,A(\tau) - A(\tau)\,A(t) \tag{49}$$

at arbitrary t and τ.

Proof: It is to be demonstrated now that eq.(34) (with arbitrary lower limit) follows from eq.(49). We calculate:

$$A(t)\int_{t_0}^{t} A(\tau)\,d\tau - \int_{t_0}^{t} A(t)A(\tau)\,d\tau = \int_{t_0}^{t} A(\tau)A(t)\,d\tau - \int_{t_0}^{t} A(\tau)\,d\tau\,A(t). \tag{50}$$

Section 3. Transformation of the Differential Equation System to a Normal Form /38

Theorem 9: By means of the transformation

$$x(t) = \Phi(t)\cdot v(t) \tag{51}$$

with $\Phi(t)$ from eq.(33), the system of differential equations (1) is transformed into the following system [Floquet's theorem, see for example (Bibl.3), Chapt.III, Sect.5, and specifically p.75]:

$$\frac{dv}{dt} = K\cdot v + b(t) \tag{52}$$

with [see eq.(32)]

$$b(t) = \Psi^{T}(t)\cdot f(t) \tag{53}$$

which has the constant matrix K according to eqs.(29) and (27) and also has a $b(t)$ periodic with P. We will call eq.(52) the normal form of our system of equations (1).

Proof: In view of eq.(32), a substitution of eq.(51) into eq.(1) and consideration of eq.(53) will yield

$$v' = \Phi^{-1}\left(A\Phi - \Phi'\right)v + b\,.$$

It then remains to demonstrate that the matrix to be applied to v is equal to K, i.e.,

$$\Phi' = A\Phi - \Phi K\,. \tag{54}$$

In fact, repeatedly taking eqs.(33) and (11) into consideration, we obtain

$$\Phi^{-1}(A\Phi - \Phi') = \Phi^{-1}\left(A\Phi - \eta' e^{-Kt} + \eta e^{-Kt} K\right) =$$
$$= \Phi^{-1}\left(A\Phi - A\eta e^{-Kt} + \Phi e^{Kt} e^{-Kt} K\right) = K. \tag{55}$$

Theorem 10: The homogeneous system /39

$$u_0' = K u_0 \tag{56}$$

conjugate to eq.(52) has no other solutions periodic with P than solutions of the form

$$u_0 = c_0 e^{\mu t} . \tag{57}$$

Naturally, an analogous statement applies also [see eq.(12)] to the system

$$\dot{u}' = -K^T u . \tag{58}$$

adjoint to eq.(56).

Proof: The system of equations (56) is resolved into the mutually independent systems

$$u_{0\nu}' = K_\nu u_{0\nu} \qquad (\nu = 1,2,\ldots,s). \tag{59}$$

Here, we then have

$$u_0 = \begin{pmatrix} u_{01} \\ u_{02} \\ \vdots \\ u_{0s} \end{pmatrix}. \tag{60}$$

Such an individual system (59) possesses the following m_ν-dimensional solution vectors

$$u_{0\nu}(t) = e^{K_\nu t} \cdot c_\nu \tag{61}$$

with an arbitrary constant vector c_ν. Because of [see eqs.(27), (23), and footnote on p.32]

$$K_\nu = \alpha_\nu E_\nu + J_\nu \text{ with } J_\nu = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \tag{62}$$

eq.(61) will be transformed into

$$u_{0\nu}(t) = e^{\alpha_\nu t E_\nu} \cdot e^{J_\nu t} \cdot c_\nu = e^{\alpha_\nu t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{m_\nu-1}}{(m_\nu-1)!} \\ & 1 & t & \cdots & \frac{t^{m_\nu-2}}{(m_\nu-2)!} \\ & & 1 & \cdots & \frac{t^{m_\nu-3}}{(m_\nu-3)!} \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} \cdot c_\nu . \tag{63}$$

Because of eq.(20) (with q = P), the vector (63) is periodic with P if and only if /40

$$\alpha_\nu = 0 \text{ and } \mathfrak{c}_\nu = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{64}$$

where $\mathfrak{c}_\nu$ has been normed. The only normed solution vector of eq.(59), periodic with P, will thus be

$$\mathfrak{w}_\nu = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{if} \quad \alpha_\nu = 0 . \tag{65}$$

The corresponding solution vector $\mathfrak{w}(t)$ of eq.(56), which is periodic with P because it is constant, will have, except for a 1 at the point

$$(\nu) = m_1 + m_2 + \ldots + m_{\nu-1} + 1 \tag{66}$$

with $\alpha_\nu = 0$, only zeros as components. The n-component solution vector of eq.(56), conjugate to eq.(65), will be denoted by $\mathfrak{w}_{(\nu)}$. This means that in eq.(60), all $\mathfrak{w}_\mu = 0$ must be used at $\mu \neq \nu$, so that $\mathfrak{w}_\nu$ will have the value [eq.(65)]:

$$\mathfrak{w}_{(\nu)} = \begin{pmatrix} 0 \\ \vdots \\ \mathfrak{w}_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{if} \quad \alpha_\nu = 0 . \tag{67}$$

This will yield the auxiliary proposition:

Lemma: The constant solution vectors of eq.(56), i.e., the vectors periodic with P, are linearly composed of the vectors (67) with $\mathfrak{w}_\nu$ from eq.(65). Analogously, the normed constant solution vectors for eq.(58) are

$$\mathfrak{v}_{[\nu]} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathfrak{v}_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{if} \quad \alpha_\nu = 0 , \tag{68}$$

with the m_ν-component vector /41

$$\mathfrak{v}_\nu = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} . \tag{69}$$

where $u_{[\nu]}$, in addition to zeros, has only one component 1 at the point

$$[\nu] = m_1 + m_2 + \ldots + m_\nu \qquad \text{with } \alpha_\nu = 0 . \tag{70}$$

Taking eqs.(30) and (31) into consideration, we obtain directly:

Theorem 11: The solution vectors $h(t)$ of eq.(3) and $\vartheta(t)$ of eq.(4), periodic with P, can be written in the form of

$$\vartheta_{(\nu)}(t) = \Phi(t) \cdot c_{(\nu)} = \bar{\varphi}_{(\nu)}(t), \tag{71}$$

and, respectively,

$$\vartheta_{[\nu]}(t) = \Delta(t) \cdot c_{[\nu]} = \bar{\varphi}_{[\nu]}(t)$$

where (ν) and $[\nu]$, respectively, have the meaning of eq.(66) and (70).

Theorem 12: For the systems of differential equations (1) and (52), we always have simultaneously the principal case or the resonance case or the exceptional case.

Proof: According to theorem 11, the solutions of eqs.(4) and (58), periodic with P, are at a one-to-one correspondence. In addition [see eqs.(5), (53), and (71)], the following applies:

$$\int_0^P \vartheta_{[\nu]}^T(\tau) f(\tau) d\tau = \int_0^P \bar{\varphi}_{[\nu]}^T(\tau) \left(\Delta^{-1}(\tau)\right)^T f(\tau) d\tau = \tag{72}$$

$$= \int_0^P c_{[\nu]}^T f(\tau) d\tau = \int_0^P b_{[\nu]}(\tau) d\tau ,$$

where the conventional notation was used. Now, the correctness of theorem 12 can be read directly from the second paragraph of Section 1.

The matrix $\Phi(t)$, defined in eq.(33), is broken up into the sum of s n-row square matrices

$$\Phi = (\varphi_{ik}) = \Phi^{(1)} + \Phi^{(2)} + \ldots + \Phi^{(s)} , \tag{73}$$

where the matrix $\Phi^{(\nu)}$, in the columns with the numbers

$(\nu), (\nu)+1, \ldots, [\nu]$ contains the vectors $\bar{\varphi}_{(\nu)}(t), \ldots, \bar{\varphi}_{[\nu]}(t)$

but otherwise only zeros*. The system of differential equations (52) decomposes

* Here, the notation of eqs.(66) and (70) has been used for the first time without the restriction $\alpha_\nu = 0$.

into the independent subsystems

$$\mathfrak{w}_\nu' = \mathfrak{A}_\nu \mathfrak{w}_\nu + \mathfrak{b}_\nu(t), \quad \nu = 1,2,\ldots,s, \tag{74}$$

where we always have the dimension m_ν; here, $\mathfrak{b}_\nu(t)$ contains only the components (ν), ..., $[\nu]$ of $\mathfrak{b}(t)$. Then, we have

$$\mathfrak{w}(t) = \begin{pmatrix} \mathfrak{w}_1(t) \\ \mathfrak{w}_2(t) \\ \vdots \\ \mathfrak{w}_s(t) \end{pmatrix} = \mathfrak{w}^{(1)}(t) + \mathfrak{w}^{(2)}(t) + \ldots + \mathfrak{w}^{(s)}(t), \tag{75}$$

if the n-component vector

$$\mathfrak{w}^{(\nu)}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathfrak{w}_\nu(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \nu = 1,2,\ldots s \tag{76}$$

is defined.

It would seem logical to introduce the vectors

$$\mathfrak{x}_*^{(\nu)}(t) = \Phi(t)\,\mathfrak{w}^{(\nu)}(t) = \Phi^{(\nu)}(t)\,\mathfrak{w}^{(\nu)}(t) = \sum_{\mu=(\nu)}^{[\nu]} \bar{\varphi}_\mu(t)\, v_\mu(t) \tag{77}$$

in which, however, all n components may differ from zero, for which reason the denotation was supplemented by an asterisk. This will yield:

Theorem 13: Each solution $\mathfrak{x}(t)$ of eq.(1) can be written in the form

$$\mathfrak{x}(t) = \sum_{\nu=1}^{s} \mathfrak{x}_*^{(\nu)}(t) \tag{78}$$

with $\mathfrak{x}_*^{(\nu)}(t)$ from eq.(77).

Lemma: If, specifically, $\mathfrak{B}$ has a diagonal form, eq.(78) with $(\nu) = [\nu] = \nu$ specializes to

$$\mathfrak{x}(t) = \sum_{\nu=1}^{n} \mathfrak{x}_*^{(\nu)}(t) = \sum_{\mu=1}^{n} \bar{\varphi}_\mu(t)\, v_\mu(t). \tag{79}$$

Resolving, in accordance with eq.(73), /44

$$A = A^{(1)} + A^{(2)} + \ldots + A^{(s)}, \tag{80}$$

as well as

$$K = K^{(1)} + K^{(2)} + \ldots + K^{(s)}, \tag{81}$$

the following theorem is obtained:

Theorem 14: The vectors $x_*^{(\nu)}(t)$, defined in eq.(77), satisfy the s systems of n differential equations each

$$x_*^{(\nu)\prime}(t) = \Omega(t)\, x_*^{(\nu)} + f^{(\nu)}(t), \quad \nu = 1,2,\ldots,s \tag{82}$$

with

$$f^{(\nu)}(t) = \Phi^{(\nu)}(t)\, b^{(\nu)}(t) = \Phi^{(\nu)}(t) A^{(\nu)T}(t)\, f(t) \tag{83}$$

under use of the above symbolism.

Proof: By differentiation of eq.(77) and with consideration of eqs.(74) and (76), we obtain

$$\begin{aligned} x_*^{(\nu)\prime} &= \Phi^{(\nu)\prime} x^{(\nu)} + \Phi^{(\nu)} x^{(\nu)\prime} \\ &= \Phi^{(\nu)\prime} x^{(\nu)} + \Phi^{(\nu)} \left(K^{(\nu)} x^{(\nu)} + b^{(\nu)} \right) \\ &= \left(\Phi^{(\nu)\prime} + \Phi^{(\nu)} K^{(\nu)} \right) x^{(\nu)} + \Phi^{(\nu)} b^{(\nu)}. \end{aligned}$$

It is readily verified that

$$\Phi^{(\nu)\prime} + \Phi^{(\nu)} K^{(\nu)} = \Omega \Phi^{(\nu)} \tag{84}$$

in which case the form of the matrix $K^{(\nu)}$ must be taken into consideration. This will further yield

$$x_*^{(\nu)\prime} = \Omega \Phi^{(\nu)} x^{(\nu)} + \Phi^{(\nu)} b^{(\nu)},$$

from which, according to eq.(77), the correctness of eq.(82) with the first /45 relation (83) is obtained. The second relation (83) can be perceived as follows: The zero columns of $\Psi^{(\nu)}$ are transferred into zero rows of $\Psi^{(\nu)T}$. Consequently, it follows from eq.(53) that

$$b^{(\nu)} = A^{(\nu)T} f. \tag{85}$$

Section 4. Discussion of the System of Differential Equations in the Normal Form

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Theorem 15: Let the matrix $\mathfrak{B}$ have a diagonal form and let $\mathfrak{x}_1, \ldots, \mathfrak{x}_\rho$ be the independent solution vectors of eq.(4) with the period P and, finally, let [eq.(72)]

$$\int_0^P \mathfrak{y}_\mu^T(\tau)\,\mathfrak{f}(\tau)\,d\tau = \int_0^P b_\mu(\tau)\,d\tau = \begin{cases} a_\mu \neq 0 & \text{for } \mu = 1,\ldots,\sigma \\ 0 & \text{for } \mu = \sigma+1,\ldots,\rho. \end{cases} \tag{86}$$

Then, for $1 \leq \mu \leq \sigma$ (partial resonance case)* each component of $\mathfrak{x}_*^{(\mu)}(t)$ [see eq.(79)] for which the corresponding component $\bar{\varphi}_\mu(t)$ does not vanish identically assumes, with increasing t, values of the order of magnitude of t; for $\mu = \sigma + 1, \ldots, \rho$ (exceptional subcase)*, all these $\mathfrak{x}_*^{(\mu)}(t)$ are periodic with P; for $\mu = \rho + 1, \ldots, n$ (partial main case)* exactly one $\mathfrak{x}_*^{(\mu)}(t)$ periodic with P exists.

Proof: Since $\mathfrak{P}(t)$ is periodic with P (theorem 5), the analogous theorems need be proved only for the $\mathfrak{v}^{(\mu)}(t)$. These have only one component $v_\mu(t)$ differing from zero which, according to eqs.(52) and (74), satisfies one equation each of the form

$$v'_\mu = \alpha_\mu v_\mu + b_\mu(t) \quad (\mu = 1,2,\ldots,n) \tag{87}$$

The general solution reads

$$v_\mu(t) = \int_0^t e^{\alpha_\mu(t-\tau)}\, b_\mu(\tau)\,d\tau + c_\mu e^{\alpha_\mu t} \tag{88}$$

with an arbitrary constant c_μ. In the case $\alpha_\mu \neq 0$ ($\mu = \rho + 1, \ldots, n$), according to eq.(88) and from the condition

$$v_\mu(t + P) = v_\mu(t)$$

the constant c_μ can be uniquely defined as

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* This case, however, must not necessarily occur.

$$c_\mu = \frac{e^{\alpha_\mu P}}{1 - e^{\alpha_\mu P}} \int_0^P e^{-\alpha_\mu \tau} b_\mu(\tau) d\tau . \tag{89}$$

In the case $\alpha_\mu = 0$, according to eq.(88), we have

$$v_\mu(t) = \int_0^t b_\mu(\tau) d\tau + c_\mu, \quad \mu = 1, 2, \ldots, \rho . \tag{90}$$

For $\mu = \sigma + 1, \ldots, \rho$, this is periodic with P at any selection of the constants c_μ. At $\mu = 1, \ldots, \sigma$, eq.(90) can be written in the following form [see eqs.(86) and (88)]:

$$v_\mu(t) = \frac{1}{P} a_\mu t + \int_0^t \left(b_\mu(\tau) - \frac{1}{P} a_\mu \right) d\tau + c_\mu, \quad \mu = 1, \ldots, \sigma \tag{91}$$

Since the integral is periodic with P, the linear increase of $v_\mu(t)$ with t follows from eq.(91) and the corresponding statement for $x_*^{(\mu)}(t)$ follows from eq.(79).

This theorem is a special case of the following:

Theorem 16: Let $\mathfrak{P}$ not necessarily be a diagonal matrix but let it be given in the form of $\mathfrak{P} = e^{\mathfrak{A}P}$ [see eqs.(29) and (27)], with the elementary components $\mathfrak{P}_\nu = e^{\mathfrak{A}_\nu P}$, where the quantities $\mathfrak{A}_\nu$ are the elementary components of the Jordan normal form of the matrix $\mathfrak{A}$ [see eq.(27) with q = P]. If $e^{\alpha_\nu P} \neq 1$ (principal subcase), then eq.(74) will have a uniquely defined solution vector $b_{(\nu)}(t)$ periodic with P, so that [see eq.(76)] the vector $x_*^{(\nu)}(t)$, defined in eq.(77), has the period P. If $e^{\alpha_\nu P} = 1$ and [see eqs.(70), (71), and (72) as well as theorem 12]

$$\int_0^P g_{[\nu]}^T(\tau) f(\tau) d\tau = \int_0^P b_{[\nu]}(\tau) d\tau = 0 \tag{92}$$

(exceptional subcase), then a one-parameter family of solution vectors $b_\nu(t)$ of eq.(74), periodic with P, will exist from which a corresponding family of vectors $x^{(\nu)}(t)$ will result. Finally, if the following relation exists in

addition to $e^{\alpha_\nu P} = 1$

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$$\int_0^P \mathfrak{y}_{[\nu]}^T(\tau)\mathfrak{f}(\tau)\,d\tau = \int_0^P b_{[\nu]}(\tau)\,d\tau = a_{[\nu]} \neq 0 \tag{93}$$

(resonance subcase), then each solution vector $\mathfrak{v}_\nu(t)$ of eq.(74) and thus also each $\mathfrak{x}^{(\nu)}(t)$, with unboundedly increasing t, will take values of the order of magnitude t^{m_ν}, provided that m_ν is the order of $\mathfrak{P}_\nu$ and $\mathfrak{R}_\nu$. (A supplement to this is contained in theorem 17.)

Proof: In the case $\alpha_\nu \neq 0$ (principal subcase), let us successively solve the system of equations (74), starting with the last equation,

$$\begin{cases} v_{[\nu]}(t) = \int_0^t e^{\alpha_\nu(t-\tau)} b_{[\nu]}(\tau)\,d\tau + c_{[\nu]} e^{\alpha_\nu t}, \\ v_\mu(t) = \int_0^t e^{\alpha_\nu(t-\tau)}\left(v_{\mu+1}(\tau) + b_\mu(\tau)\right)d\tau + c_\mu e^{\alpha_\nu t} \text{ for } \mu = [\nu]-1, [\nu]-2, \ldots, (\nu). \end{cases} \tag{94}$$

The condition $v_\mu(t + P) = v_\mu(t)$, analogous to eq.(89), will successively yield

$$\begin{cases} c_{[\nu]} = \dfrac{e^{\alpha_\nu P}}{1 - e^{\alpha_\nu P}} \int_0^P e^{-\alpha_\nu \tau} b_{[\nu]}(\tau)\,d\tau, \\ c_\mu = \dfrac{e^{\alpha_\nu P}}{1 - e^{\alpha_\nu P}} \int_0^P e^{-\alpha_\nu \tau}\left[v_{\mu+1}(\tau) + b_\mu(\tau)\right]d\tau, \quad \mu = [\nu]-1, \ldots, (\nu). \end{cases} \tag{95}$$

Thus, the vector $\mathfrak{v}_\nu(t)$ periodic with P and thus also $\mathfrak{x}_*^{(\nu)}(t)$ is uniquely defined in this case as well*. If $\alpha_\nu = 0$, then

$$\mathfrak{R}_\nu = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{[\nu]} \end{pmatrix} \text{ and } a_{[\nu]} = \int_0^P b_{[\nu]}(\tau)\,d\tau \tag{96}$$

* If $\Re(\alpha_\nu) \neq 0$, the integrals in eq.(94) with $t_0 = \operatorname{sign} \Re(\alpha_\nu) \cdot \infty$ instead of 0 as lower limit, will automatically have the period P, as can be readily verified. In that case, the last summands in eq.(94) must be omitted [$c_\mu = 0$ for $\mu = [\nu], [\nu] - 1, \ldots, (\nu)$].

[see eq.(86)] is used for decomposing the vector in eq.(74) /49

$$b_\nu(t) = \frac{1}{P} a_\nu + b_\nu^*(t), \tag{97}$$

where

$$\int_0^P b^*_{[\nu]}(\tau)\,d\tau = 0 \tag{98}$$

Let

$$\omega_\nu(t) = \eta_\nu(t) + \varphi^*_\nu(t), \tag{99}$$

where, because of theorem 10, it still can be stipulated, for example, that the first component $q_{(\nu)}(t)$ of $q_\nu(t)$ satisfies the condition

$$\int_0^P q_{(\nu)}(t)\,dt = 0 \tag{100}$$

Then, eq.(74) is equivalent to the two systems

$$\eta'_\nu(t) = \alpha_\nu \eta_\nu(t) + \frac{1}{P} a_\nu \tag{101}$$

and

$$\varphi^{*\prime}_\nu(t) = \alpha_\nu \varphi^*_\nu(t) + \beta^*_\nu(t). \tag{102}$$

Solution of eq.(102) in a manner analogous to that used for eq.(94), will yield

$$\begin{cases} v^*_{[\nu]}(t) = \int_0^t b^*_{[\nu]}(\tau)\,d\tau + c^*_{[\nu]} \\ v^*_\mu(t) = \int_0^t \left(v^*_{\mu+1}(\tau) + b^*_\mu(\tau)\right) d\tau + c^*_\mu, \quad \mu = [\nu]-1, \ldots, (\nu). \end{cases} \tag{103}$$

The condition $v^*(t + P) = v^*(t)$, because of the fact that $b_\mu(\tau) = b^*_\mu(\tau)$ is valid for $\mu = [\nu] - 1, \ldots, (\nu)$, will yield

$$\int_0^P b^*_{[\nu]}(\tau)\,d\tau = 0 \quad \text{and} \quad \int_0^P \left(v^*_{\mu+1}(\tau) + b_\mu(\tau)\right) d\tau = 0. \tag{104}$$

Because of eq.(98), the first condition is automatically satisfied. Conversely, the remaining conditions (104) successively yield unique values for the constants $c^*_{[\nu]}$, $c^*_{[\nu]-1}$, ..., $c^*_{(\nu)+1}$, whereas $c^*_{(\nu)}$ remains completely arbitrary. This /50 means that eq.(102) has a one-parameter family of solutions $v^*_\nu(t)$ periodic with

P. If now, in addition to $\alpha_\nu = 0$, also eq.(92) is satisfied (exceptional subcase), then a_ν of eq.(96) is the null vector and eq.(102) has the zero vector as the only solution vector periodic with P which satisfies eq.(100). This means that the above calculated one-parameter family of solutions $v_\nu^*(t)$, periodic with P and having $c_{(\nu)}^*$ as parameter, furnishes the only solutions $v_\nu(t) = v_\nu^*(t)$ of eq.(74), periodic with P. From this, the quantities $x_*^{(\nu)}(t)$, periodic with P, can be calculated according to eq.(77). Conversely, if eq.(93) coexists with $\alpha_\nu = 0$, the system (101)

$$\eta'_{[\nu]} = \frac{1}{P} a_{[\nu]} \quad , \quad \eta'_\mu = \eta_{\mu+1} \quad , \quad \mu = [\nu]-1, \ldots, (\nu) \tag{105}$$

is solved in the form of

$$\begin{aligned} q_\sigma(t) = {} & \text{Polynomial of the degree } m_1 + m_2 + \ldots + m_\nu + 1 - \sigma \\ & = [\nu] + 1 - \sigma \quad (\sigma = (\nu), (\nu)+1, \ldots, [\nu]) \end{aligned} \tag{106}$$

with the highest coefficient $\frac{\frac{1}{P} a_{[\nu]}}{([\nu]+1-\sigma)!}$. Consequently, specifically $q_{(\nu)}(t)$ will be a polynomial of the degree m_ν. Since, because of eq.(98), the terms $v_\sigma^*(t)$ in eq.(103) increase at most like $t^{[\nu]-\sigma}$, meaning that $v_{(\nu)}^*(t)$ at most will increase like $t^{m_\nu - 1}$, each solution $v_\nu(t)$, with increasing t, will take values of the order of t^{m_ν} in accordance with eq.(99). The same statement then also applies to the vectors $x_*^{(\nu)}(t)$ to be calculated from eq.(77) since, according to eq.(30), $\bar{\varphi}_{(\nu)}(t)$ cannot be the null vector.

Without further proof, these same considerations show directly:

Lemma: The components $v_\sigma(t)$ ($\sigma = (\nu), (\nu)+1, \ldots, [\nu]$), in the resonance subcase, take values of the order of $t^{[\nu]+1-\sigma}$ with increasing t. The same statement applies to the vectorial component summands $\bar{\varphi}_\sigma(t)v_\sigma(t)$, occurring in $x_*^{(\nu)}(t)$ in accordance with eq.(77), for each scalar component for which the corresponding component of $\bar{\varphi}_\sigma(t)$ differs from zero (see theorem (17). /51

Theorem 17: In the resonance subcase, each component $x_{*\sigma}^{(\nu)}(t)$ ($\sigma = 1, 2, \ldots, n$) of $x_*^{(\nu)}(t)$ from eq.(77) takes arbitrarily large values with increasing t, unless, for example for $\sigma = \rho$, the components $\varphi_{0\rho\mu}(\mu = (\nu), (\nu)+1, \ldots, [\nu])$ of the ρ^{th} row vector of Φ [see eq.(33)] all vanish; in this case, we have $x_{*\rho}^{(\nu)}(t) \equiv 0$. More accurately, it can be stated: If we do not have $x_{*\rho}^{(\nu)}(t) \equiv 0$ (see above), then $x_{*\rho}^{(\nu)}(t)$ ($\rho = 1, 2, \ldots, n$), with increasing t, will take values of the order of $t^{m_\nu - \lambda}$ provided that in the ρ^{th} row of $\Phi^{(\nu)}$ [see eq.(73)], the quantity $\varphi_{0\rho,(\nu)+\lambda}$ is the first component differing from zero.

Proof: According to eq.(77), (99), and (106), we can write

$$x_*^{(\nu)}(t) = \Phi^{(\nu)}(t)\left(q^{(\nu)}(t) + v^{*(\nu)}(t)\right),$$

if $q^{(\nu)}(t)$ or $v^{*(\nu)}(t)$ represent the n-component vectors supplemented by zeros from $q_\nu(t)$ and $v_\nu^*(t)$. Rather than by eq.(100), the resolution of eq.(99) is now restricted by the stipulation that $v_\nu^*(t)$ is to be periodic with P and normed. In addition, we have

$$x_*^{(\nu)}(t) = \Phi^{(\nu)} v^{*(\nu)} + \sum_{\sigma=(\nu)}^{[\nu]} \bar{\varphi}_\sigma q_\sigma \qquad (107)$$

[see eq.(33)]. In this case, $\bar{\varphi}_{(\nu)}(t), \ldots, \bar{\varphi}_{[\nu]}(t)$ are linearly independent vectors with the period P while $q_\sigma(t)$, according to eq.(106), is a polynomial in t of the degree [note: $\sigma = (\nu) + \lambda$]:

$$[\nu] + 1 - \sigma = m_\nu - \lambda. \qquad (108)$$

Thus, the ρ^{th} component of eq.(107), taking eq.(105) into consideration, will become

$$x_{*\rho}^{(\nu)}(t) = \sum_{\sigma=(\nu)}^{[\nu]} \varphi_{\rho\sigma}(t)\, v_\sigma^*(t) + \sum_{\sigma=(\nu)}^{[\nu]} \varphi_{\rho\sigma}(t)\, q_\sigma(t). \qquad (109)$$

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Since the first sum in eq.(109) is periodic with P, it will remain finite for all t. If $\varphi_{0\rho,(\nu)+\lambda}$ is the first nonvanishing coefficient in the second sum, then $x_{*0}^{(\nu)}(t)$ will take values of the order of magnitude $t^{m_\nu - \lambda}$ with increasing t.

We still note the following (slightly weaker) alternative:

Alternative: Either $x_{*0}^{(\nu)}(t)$ assumes arbitrarily large values with increasing t or else $x_{*0}^{(\nu)}(t) \equiv 0$ is valid.

Note: If, corresponding to eq.(73),

$$\mathfrak{y} = \mathfrak{y}^{(1)} + \mathfrak{y}^{(2)} + \cdots + \mathfrak{y}^{(\alpha)} \tag{110}$$

is defined, then from the vanishing of all elements of the ρ^{th} row of $\Phi^{(\nu)}$, because of eqs.(30) or (33), the same statement follows for $\mathfrak{y}^{(\nu)}$ and vice versa.

Section 5. Examples /53

Example 1: (Re theorem 15, resonance case and exceptional case, and theorem 8).

Let us consider the system of differential equations

$$\begin{cases} x_1' = x_1 \cos t + x_2 \sin t + f_1(t) \\ x_2' = -x_1 \sin t + x_2 \cos t + f_2(t) \end{cases} \tag{111}$$

under the following two assumptions:

$$\begin{cases} f_1(t) = e^{\sin t} \cos(1 - \cos t) \\ f_2(t) = -e^{\sin t} \sin(1 - \cos t) \end{cases} \tag{111a}$$

and

$$\begin{cases} f_1(t) = 2e^{\sin t} \sin t \sin(1 - \cos t) \\ f_2(t) = -2e^{\sin t} \sin t \cos(1 - \cos t). \end{cases} \tag{111b}$$

In eq.(111), we have

$$\mathfrak{A}(t) = \left(\begin{array}{c|c} \cos t & \sin t \\ \hline -\sin t & \cos t \end{array}\right) = \cos t\, \mathfrak{E} + \sin t\, \mathfrak{J} \qquad *(2.1)$$

with

$$\mathfrak{J} = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right). \tag{112}$$

* The numerals in front of the period refer to numbers in the main text while the numerals behind the period designate the number of the example involved.

On the basis of this latter presentation (2.1) the validity of eq.(49) can be directly verified. Therefore, at $P = 2\pi$, the quantity $\mathfrak{R}^{**}$ from eq.(35) will /54 be the zero matrix, i.e.,

$$\mathfrak{R} = \mathfrak{E} \tag{36.1}$$

and

$$\Phi(t) = \mathfrak{U}(t) \tag{30.1}$$

as well as

$$\mathfrak{L}(t) = \mathfrak{Z}(t). \tag{31.1}$$

In addition, eq.(37) must be used as the solution matrix of eq.(3), because of eq.(49) (see theorem 6). Taking eq.(2.1) into consideration, this will yield

$$\begin{aligned}\mathfrak{U}(t) &= e^{\int_0^t (\cos\tau\,\mathfrak{E} + \sin\tau\,\mathfrak{J})\,d\tau} = e^{\sin t\,\mathfrak{E}}\cdot e^{(1-\cos t)\mathfrak{J}} \\ &= e^{\sin t}\cdot\mathfrak{E}\left[\mathfrak{E} + (1-\cos t)\mathfrak{J} - \frac{(1-\cos t)^2}{2!}\mathfrak{E} - \frac{(1-\cos t)^3}{3!}\mathfrak{J} + \cdots\right] \\ &= e^{\sin t}\cdot\mathfrak{E}\left[\cos(1-\cos t)\,\mathfrak{E} + \sin(1-\cos t)\,\mathfrak{J}\right] \\ &= \begin{pmatrix} \cos(1-\cos t) & \sin(1-\cos t) \\ -\sin(1-\cos t) & \cos(1-\cos t)\end{pmatrix}\begin{pmatrix} e^{\sin t} & 0 \\ 0 & e^{\sin t}\end{pmatrix}.\end{aligned} \tag{37.1}$$

Therefore, it follows that*

$$\mathfrak{Z}(t) = \begin{pmatrix} \cos(1-\cos t) & \sin(1-\cos t) \\ -\sin(1-\cos t) & \cos(1-\cos t)\end{pmatrix}\begin{pmatrix} e^{-\sin t} & 0 \\ 0 & e^{-\sin t}\end{pmatrix}. \tag{12.1}$$

All solutions of eq.(4) thus will have the period $P = 2\pi$.

In the case (111a), the resonance case is present since it is calculated that

$$\begin{cases} \int_0^{2\pi} \mathfrak{Z}_1^{\top}(\tau)\,\mathfrak{B}(\tau)\,d\tau = \int_0^{2\pi} 1\cdot d\tau = 2\pi & \text{(resonance subcase)} \\ \int_0^{2\pi} \mathfrak{Z}_2^{\top}(\tau)\,\mathfrak{B}(\tau)\,d\tau = \int_0^{2\pi} 0\cdot d\tau = 0 & \text{(exceptional subcase)}\end{cases} \tag{86.1a}$$

* By substituting eqs.(37.1) and (12.1) into eq.(10), it is possible to confirm eq.(36.1) as control.

Taking eqs.(53), (75), (76), and (31.1) into consideration, we obtain /55

$$v_\nu(t) = \int b_\nu(\tau)\,d\tau = \int j_\nu^T(\tau) f(\tau)\,d\tau\,, \tag{74.1}$$

and thus, under consideration of eq.(30.1),

$$z^{(\nu)}(t) = y_\nu(t)\, v_\nu(t) = y_\nu(t)\left[\int j_\nu^T(\tau) f(\tau)\,d\tau + c_\nu\right]. \tag{79.1}$$

Calculating out, eq.(37.1) in view of eq.(86.1a) will yield

$$\begin{cases} z^{(1)}(t) = \begin{pmatrix} \cos(1-\cos t) \\ -\sin(1-\cos t) \end{pmatrix} e^{\sin t}\ (t + c_1), \\ z^{(2)}(t) = \begin{pmatrix} \sin(1-\cos t) \\ \cos(1-\cos t) \end{pmatrix} e^{\sin t}\, c_2\,. \end{cases} \tag{113}$$

In accordance with theorem 15, each component of $z^{(1)}(t)$, with increasing t, will again and again take values of the order of t, while $z^{(2)}(t)$, as a vector periodic with 2π, will remain restricted. The general solution of eq.(111) then becomes

$$z(t) = z^{(1)}(t) + z^{(2)}(t)\,. \tag{78.1}$$

In the case (111b), the exceptional case exists because of

$$\begin{cases} \int_0^{2\pi} j_1^T(\tau) f(\tau)\,d\tau = 2\int_0^{2\pi} \sin\tau \sin(2-2\cos\tau)\,d\tau = -\cos(2-2\cos\tau)\Big|_0^{2\pi} = 0, \\ \int_0^{2\pi} j_2^T(\tau) f(\tau)\,d\tau = -2\int_0^{2\pi} \sin\tau\cos(2-2\cos\tau)\,d\tau = -\sin(2-2\cos\tau)\Big|_0^{2\pi} = 0. \end{cases} \tag{86.1b}$$

Analogous to eq.(113), calculation yields

$$\begin{cases} z^{(1)}(t) = \begin{pmatrix} \cos(1-\cos t) \\ -\sin(1-\cos t) \end{pmatrix} e^{\sin t}\ \left[1-\cos(2-2\cos t)+c_1\right] \\ z^{(2)}(t) = \begin{pmatrix} \sin(1-\cos t) \\ \cos(1-\cos t) \end{pmatrix} e^{\sin t}\left[-\sin(2-2\cdot\cos t)+c_2\right]. \end{cases} \tag{114}$$

The general solution of eq.(111), composed in accordance with eq.(78.1), thus

has the period 2π.

Example 2. (Re theorem 15, principal case, and theorem 8). /56

The system

$$\begin{cases} x_1' = (1+\cos t)x_1 + (2+\sin t)x_2 + e^{\sin t}\cos\left((1+2t)-\cos t\right) \\ x_2' = -(2+\sin t)x_1 + (1+\cos t)x_2 - e^{\sin t}\sin\left((1+2t)-\cos t\right) \end{cases} \tag{115}$$

possesses the matrix

$$A(t)\left(\begin{array}{c|c} 1+\cos t & 2+\sin t \\ \hline -2-\sin t & 1+\cos t \end{array}\right) = (1+\cos t)\mathcal{E} + (2+\sin t)\mathcal{J} \tag{2.2}$$

with $\mathcal{J}$ from eq.(112) and satisfies the condition (49). As in the above example, we find

$$\mathcal{A}^{**} = \mathcal{E} + 2\mathcal{J}. \tag{35.2}$$

so that

$$\Phi = e^{2\pi(\mathcal{E}+2\mathcal{J})} = e^{2\pi}\cdot\mathcal{E}\cdot e^{4\pi\mathcal{J}} = e^{2\pi}\mathcal{E}, \tag{36.2}$$

since derivation is made by the method of power series which consists in expanding the exponential function

$$e^{4\pi\mathcal{J}} = \mathcal{E}$$

Consequently, in the diagonal matrix $\mathfrak{B}$, we have [see eq.(16)]

$$\lambda_1 = \lambda_2 = e^{2\pi}.$$

Thus, the principal case is involved here. In addition, we calculate

$$\begin{aligned} \mathcal{U}(t) &= e^{(t+\sin t)\mathcal{E}}\cdot e^{(1+2t-\cos t)\mathcal{J}} \\ &= \left(\begin{array}{c|c} \cos(1+2t-\cos t) & \sin(1+2t-\cos t) \\ \hline -\sin(1+2t-\cos t) & \cos(1+2t-\cos t) \end{array}\right)\left(\begin{array}{c|c} e^{t+\sin t} & 0 \\ \hline 0 & e^{t-\sin t} \end{array}\right), \end{aligned} \tag{37.2}$$

from which, by means of eq.(10), we again obtain eq.(36.2). Analogous to /57 eq.(113), we find

$$\begin{cases} z^{(1)}(t) = \begin{pmatrix} \cos(1+2t-\cos t) \\ -\sin(1+2t-\cos t) \end{pmatrix} e^{t+\sin t}\cdot(-e^{-t}+1+c_1), \\ z^{(2)}(t) = \begin{pmatrix} \sin(1+2t-\cos t) \\ \cos(1+2t-\cos t) \end{pmatrix} e^{t+\sin t}\cdot c_2 . \end{cases} \tag{116}$$

With $c_1 = -1$, the quantity $\xi^{(1)}(t)$ becomes periodic with 2π while at $c_2 = 0$ the vector $\xi^{(2)}(t)$ and thus also the total vector (78) becomes trivially periodic.

Example 3. (Re theorem 16, resonance and exceptional case, and theorem 8).

Consider the system of differential equations

$$\begin{cases} x_1' = -x_1 \sin t + x_2 \left[1 + (\sin t + \cos t) e^{\sin t - \cos t}\right] + f_1(t) \\ x_2' = -x_2 \sin t + f_2(t) \end{cases} \tag{117}$$

with

$$\begin{cases} f_1(t) = e^{\sin t} - 1 \\ f_2(t) = e^{\cos t} - 1 \end{cases} \tag{117a}$$

and, respectively,

$$\begin{cases} f_1(t) = e^{\cos t} - 1 \\ f_2(t) = 0. \end{cases} \tag{117b}$$

The coefficient matrix becomes

$$\mathfrak{A}(t) = \left(\begin{array}{c|c} -\sin t & 1 + (\sin t + \cos t)\, e^{\sin t - \cos t} \\ \hline 0 & -\sin t \end{array}\right) \tag{118}$$

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$$= -\sin t\, \mathfrak{E} + \left[1 + (\sin t + \cos t) e^{\sin t - \cos t}\right] \mathfrak{H},$$

if we define

$$\mathfrak{H} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{119}$$

Here, $\mathfrak{A}(t)$ satisfies the assumption (49). We can then calculate directly

$$\mathfrak{K}^{**} = \mathfrak{H} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{35.3}$$

so that, because of $\mathfrak{H}^2 = 0$,

$$\mathfrak{M} = e^{2\pi \mathfrak{H}} = \mathfrak{E} + 2\pi \mathfrak{H} = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix}. \tag{36.3}$$

Consequently, $\mathfrak{M}$ is an elementary component of the rank $m = 2$, with the eigenvalue $\lambda = 1$. Further, we obtain

$$\begin{aligned} \mathfrak{y}(t) &= e^{(\cos t-1)}\mathfrak{k} + \left(e^{\sin t-\cos t}e^{-1}+t\right)\mathfrak{y} \\ &= e^{\cos t-1}\mathfrak{k}\cdot\left[\mathfrak{k}+\left(e^{\sin t-\cos t}e^{-1}+t\right)\mathfrak{y}\right] \\ &= e^{\cos t-1}\mathfrak{k}+\left[e^{\sin t-1}+te^{\cos t-1}-e^{\cos t-2}\right]\mathfrak{y} \\ &= \left(\begin{array}{c|c} e^{\cos t-1} & e^{\sin t-1}-e^{\cos t-2}+te^{\cos t-1} \\ \hline 0 & e^{\cos t-1} \end{array}\right) \\ &= \Phi(t)\mathfrak{k}^{**}t \end{aligned} \tag{37.3}$$

with

$$\Phi(t) = \left(\begin{array}{c|c} e^{\cos t-1} & e^{\sin t-1}-e^{\cos t-2} \\ \hline 0 & e^{\cos t-1} \end{array}\right), \tag{120}$$

where eqs.(30), (35.3), and (36.3) have been taken into consideration. By means of

$$\mathfrak{r}(t) = \Phi(t)\cdot\mathfrak{u}(t) \tag{51.3}$$

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the following system of equations is obtained:

$$\frac{d\mathfrak{w}}{dt} = \mathfrak{k}^{**}\cdot\mathfrak{u}+\mathfrak{b}(t) \tag{52.3}$$

with [see eq.(32)]

$$\mathfrak{b}(t) = \Phi^{-1}(t)\mathfrak{f}(t). \tag{53.3}$$

In the case of eq.(117a) and in accordance with eqs.(120) and (117a), we have

$$\begin{aligned} \mathfrak{b}(t) &= \left(\begin{array}{c|c} e^{-\cos t+1} & -e^{\sin t-2\cos t+1}+e^{-\cos t} \\ \hline 0 & e^{-\cos t+1} \end{array}\right)\left(\begin{array}{c} e^{\sin t-1} \\ e^{\cos t-1} \end{array}\right) = \\ &= \left(\begin{array}{c} e^{-1} \\ 1 \end{array}\right). \end{aligned} \tag{53.3a}$$

Because of

$$\int_0^{2\pi} b_2(\tau)\,d\tau = 2\pi \tag{92.3a}$$

the resonance case is present.

From eq.(52.3) [see also eq.(35.3)], we then obtain

$$\begin{aligned} v_1' &= v_2 + e^{-1} \\ v_2' &= 1\ , \quad v_2 = t + c_2\ , \\ v_1 &= \frac{t^2}{2} + c_2 t + e^{-1}t + c_1\ , \end{aligned}$$

so that, according to eq.(51.3),

$$\mathfrak{x}(t)=\begin{pmatrix} e^{\cos t-1}\left(\frac{t^2}{2}+c_2\,t+e^{-1}t+c_1\right)+\left(e^{\sin t-1}-e^{\cos t-2}\right)(t+c_2) \\ e^{\cos t-1}\,(t+c_2)\end{pmatrix}. \tag{121a}$$

In accordance with theorem 16, $\mathfrak{x}(t)$ takes values of the order of magnitude t^2; in accordance with the lemma of theorem 16, $v_1(t)$ and thus also $x_1(t)$ again and again are of the order of magnitude t^2, while $v_2(t)$ and $x_2(t)$ are of the order of magnitude t. /60

Analogously, in the case of eq.(117b), we obtain

$$\mathfrak{b}(t)=\begin{pmatrix}1\\0\end{pmatrix}, \tag{53.3b}$$

$$\int_0^{2\pi} b_1(\tau)\,d\tau=0\ , \tag{92.3b}$$

so that the exceptional case is involved here. From eq.(52.3) we obtain

$$v_1'=v_2+1$$

$$v_2'=0\ ,\quad v_2=c_2\ ,$$

$$v_1=(1+c_2)t+c_1\ ,$$

and thus also, in accordance with eqs.(51.3) and (120),

$$\mathfrak{x}(t)=\begin{pmatrix}\left[te^{\cos t-1}+e^{\sin t-1}-e^{\cos t-2}\right]c_2+e^{\cos t-1}(c_1+t)\\ c_2\,e^{\cos t-1}\end{pmatrix}. \tag{121b}$$

Obviously,

$$\mathfrak{x}(t+2\pi)=\mathfrak{x}(t)\ \text{is valid for}\ c_2=-1$$

Example 4. [Re theorem 15, resonance and exceptional case, at general $\mathfrak{A}(t)$].

Consider the system /61

$$\begin{cases} x_1'=(1+\sin t)\,x_1+\sin t\,x_2+f_1(t)\\ x_2'=\sin t\,x_2+f_2(t)\end{cases} \tag{122}$$

with

$$\begin{cases} f_1(t) = -\frac{1}{2} e^{1-\cos t}(\sin t + \cos t) \\ f_2(t) = e^{1-\cos t} \end{cases} \tag{122a}$$

and, respectively,

$$\begin{cases} f_1(t) = e^{1-\cos t} \\ f_2(t) = 0 . \end{cases} \tag{122b}$$

The coefficient matrix

$$\mathfrak{A}(t) = \left(\begin{array}{c|c} 1+\sin t & \sin t \\ \hline 0 & \sin t \end{array}\right) \tag{123}$$

is not commutative either with $\mathfrak{A}(\tau)$ or with $\int_0^t \mathfrak{A}(\tau)d\tau$, as can be readily checked by calculation. From the second equation of the system (3) we find, together with eq.(123),

$$y_2(t) = c_2 e^{1-\cos t},$$

and thus from the first equation of the system (3),

$$y_1(t) = e^t + 1 - \cos t\left(c_1 + c_2 \int_0^t \sin\tau\, e^{\cos\tau}\, d\tau\right) =$$

$$= c_1 e^t e^{1-\cos t} + c_2 e^t e^{1-\cos t}\left(-e^{-t}\frac{\cos t + \sin t}{2} + \frac{1}{2}\right),$$

so that we can select

$$\mathfrak{Y} = e^{1-\cos t}\left(\begin{array}{c|c} e^t & -\frac{1}{2}(\sin t + \cos t) \\ \hline 0 & 1 \end{array}\right). \tag{7.4}$$

Hence 162

$$\mathfrak{P} = \mathfrak{Y}^{-1}(0)\cdot\mathfrak{Y}(2\pi) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{2\pi} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2\pi} & 0 \\ 0 & 1 \end{pmatrix}. \tag{10.4}$$

The matrix $\mathfrak{P}$ is partitioned into the two elementary components $\mathfrak{P}_1$ with the eigenvalue $\lambda_1 = e^{2\pi}$ and $\mathfrak{P}_2$ with $\lambda_2 = 1$.

Further, we calculate

$$\mathfrak{Z}\cdot\mathfrak{y}^{-1^T} = e^{\cos t - 1}\left(\begin{array}{c|c} e^{-t} & 0 \\ \hline \frac{1}{2}e^{-t}(\sin t + \cos t & 1 \end{array}\right). \tag{12.4}$$

In the case of eq.(122a), because of

$$\int_0^{2\pi} \mathfrak{z}_2^T(\tau)\,\mathfrak{A}(\tau)\,d\tau = 2\pi \tag{86.4a}$$

the resonance subcase is present for $\mathfrak{P}_2$ and the principal case for $\mathfrak{P}_1$. We then use the method of variation of the constants* for further calculation. In the argument

$$\mathfrak{x}(t) = \mathfrak{x}_1^*(t) + \mathfrak{x}_2^*(t) \tag{9.4}$$

we then have /63

$$\begin{cases} \mathfrak{x}_1^*(t) = e^{1-\cos t}\begin{pmatrix} e^t \\ 0 \end{pmatrix} c_{01} \\ \mathfrak{x}_2^*(t) = e^{1-\cos t}\begin{pmatrix} -\frac{1}{2}(\sin t + \cos t) \\ 1 \end{pmatrix}(t + c_{02}). \end{cases} \tag{12.4}$$

The quantity $\mathfrak{x}_2^*(t)$ again and again assumes values of the order of magnitude t while $\mathfrak{x}_1^*(t)$, on selection of $c_{01} = 0$, is periodic with 2π.

In the case of eq.(122b), because of

$$\int_0^{2\pi} \mathfrak{z}_2^T(\tau)\,\mathfrak{A}(\tau)\,d\tau = 0 \tag{86.4b}$$

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* On substituting eq.(9), under consideration of eq.(8), into eq.(1), we will have

$$\mathfrak{x}' = \mathfrak{y}'\mathfrak{x} + \mathfrak{y}\mathfrak{x}' = \mathfrak{A}\mathfrak{x} + \mathfrak{A} = \mathfrak{A}\mathfrak{y}\mathfrak{x} + \mathfrak{A};$$

i.e., according to eq.(12)

this will yield

$$\mathfrak{x}' = \mathfrak{y}^{-1}\mathfrak{A} = \mathfrak{Z}^T\mathfrak{A},$$

$$\mathfrak{x}(t) = \int_0^t \mathfrak{Z}^T(\tau)\,\mathfrak{A}(\tau)\,d\tau + \mathfrak{x}_0.$$

For the summands of eq.(9) in view of eq.(8), this furnishes

$$\mathfrak{x}_\nu^*(t) = \mathfrak{y}_\nu(t)\left(\int_0^t \mathfrak{z}_\nu^T(\tau)\,\mathfrak{A}(\tau)\,d\tau + c_{0\nu}\right).$$

the exceptional subcase exists for $\mathfrak{R}_2$ and the principal case for $\mathfrak{R}_1$. As above, eq.(9.4) is obtained with

$$\begin{cases} \tau_1^*(t) = e^t - \cos t \begin{pmatrix} e^t \\ 0 \end{pmatrix} \cdot \left(-e^{-t} + c_{01}\right) \\ \tau_2^*(t) = e^t - \cos t \begin{pmatrix} -\frac{1}{2}(\sin t + \cos t) \\ 1 \end{pmatrix} \cdot c_{02} \,. \end{cases} \tag{125}$$

Here, $\tau_2^*(t)$, for any c_{02}, has the period 2π while $\tau_1^*(t)$ has this only for $c_{01} = 0$.

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PART IV

THE RESONANCE CASE IN LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE n^{th} ORDER

Section 1. General Considerations

In an earlier paper (Bibl.1), we investigated the resonance case in systems of n linear ordinary differential equations of the first order with periodic coefficients. Here, this theory will be applied to linear ordinary differential equations of the n^{th} order with periodic coefficients. Consequently, let the differential equation

$$L[x] \equiv x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = f(t), \tag{1}$$

be given in which, for all coefficient functions and for the function f(t), reality, continuity, and periodicity with the period P are assumed:

$$a_\mu(t + P) = a_\mu(t) \ (\mu = 1, \ldots, n), \quad f(t + P) = f(t). \tag{2}$$

Using the notations

$$_1x = x, \ _2x = x', \ldots, \ _nx = x^{(n-1)} \tag{3}$$

the differential equation (1) is transformed into the system

$$\tau' = \mathfrak{A}(t) + \mathfrak{f}(t) \tag{4}$$

with

$$\tau = \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix}, \quad \mathfrak{A}(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n(t), & -a_{n-1}(t), & \cdots, & -a_2(t), & -a_1(t) \end{bmatrix}, \quad \mathfrak{f}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}. \tag{5}$$

To the homogeneous differential equation conjugate to eq.(1)

$$L[y] \equiv y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = 0 \tag{6}$$

the following system of differential equations will then correspond

$$\eta' = \mathfrak{A}(t)\eta \tag{7}$$

with /68

$$y = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} . \tag{8}$$

The homogeneous differential equation, adjoint to eq.(6),

$$\bar{L}[z] = (-1)^n z^{(n)} + (-1)^{n-1}(a_1(t)z)^{(n-1)} + \dots + a_n(t)z = 0 \tag{9}$$

when using the notations

$$\begin{cases} n^z = z \\ n-1^z = -n^{z'} + a_1 n^z = a_1 z - z' \\ n-2^z = -n-1^{z'} + a_2 n^z = a_2 z - (a_1 z)' + z'' \\ n-3^z = -n-2^{z'} + a_3 n^z = a_3 z - (a_2 z)' + (a_1 z)'' - z''' \\ \quad \vdots \\ 1^z = -{}_2z' + a_{n-1} n^z = a_{n-1} z - (a_{n-2} z)' + (a_{n-3} z)'' - \dots + (-1)^n z^{(n-1)} \\ 0 = -{}_1z' + a_n n^z = \bar{L}[z] \end{cases} \tag{10}$$

will be transformed into the system of differential equations

$$\eta' = -A^T(t)\eta \tag{11}$$

with

$$\eta = \begin{pmatrix} {}_1z \\ {}_2z \\ \vdots \\ {}_nz \end{pmatrix} \tag{12}$$

In another paper [Bibl.2, eq.(17)], the following was demonstrated:

If $\mathfrak{y}(t)$ is a fundamental solution matrix of eq.(7), then

$$Z(t) = (\mathfrak{y}^{-1}(t))^T = (\eta_1, \eta_2, \dots, \eta_n) \tag{13}$$

is a fundamental solution matrix of eq.(11). According to another paper /69 [(Bibl.1), eq.(30)], a fundamental solution matrix $\mathfrak{y}(t)$ of the following form*

* For abbreviation, we will later write ${}_1\varphi_i(t) = \varphi_i(t)$ and analogously in the first component of other vectors.

$$\eta(t) = \Phi(t) \cdot e^{\mathfrak{R}t}, \quad \Phi(t) = (\bar{\varphi}_1(t), \ldots, \bar{\varphi}_n(t)), \quad \bar{\varphi}_\mu(t) = \begin{pmatrix} {}^1\varphi_\mu(t) \\ {}^2\varphi_\mu(t) \\ \vdots \\ {}^n\varphi_\mu(t) \end{pmatrix} \tag{14}$$

exists, having a matrix $\Phi(t)$ periodic with P and a constant matrix $\mathfrak{R}$ which latter is written in the Jordan normal form [see (Bibl.1), eqs.(27), (28), (29)], from which, for the fundamental solution matrix (13), the following presentation is obtained [see (Bibl.1) eqs.(31) and (32)]:

$$\Psi(t) = \left(\Phi^{-1}(t)\right)^T \cdot e^{-\mathfrak{R}^T t} = L(t) \cdot e^{-\mathfrak{R}^T t} \text{ with } L = (\bar{\psi}_1, \ldots, \bar{\psi}_n) = \left(\Phi^{-1}(t)\right)^T \tag{15}$$

We will treat the general case [see (Bibl.1) eq.(27)] that the matrix $\mathfrak{R}$ is partitioned into s elementary components

$$\mathfrak{K}_\nu = \begin{pmatrix} \alpha_\nu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_\nu \end{pmatrix} \quad , \nu = 1, 2, \ldots, s \tag{16}$$

with the orders m_ν, in which case the following is assumed to be valid

$$\alpha_\nu \begin{cases} = 0 & \text{for } \nu = 1, \ldots, \rho \\ \neq 0 & \text{for } \nu = \rho + 1, \ldots, s \end{cases} \tag{17}$$

($\rho = 0$ and $\rho = s$ is admitted).

For the general solution x(t) of eq.(1), using the method of variation of the constants [see (Bibl.1), eq.(9)] and the footnote on p.58, we obtain

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$$x(t) = \sum_{\mu=1}^{n} y_\mu(t) \cdot \int z_\mu(t)\, f(t)\, dt \tag{18}$$

[see eqs.(5), (10), and (13)] which can be resolved, in the form of*

$$x(t) = \sum_{\nu=1}^{s} {}^\nu x(t) \tag{19}$$

into the components

$${}^\nu x(t) = \sum_{\mu=\nu(\nu)}^{\nu[\nu]} y_\mu(t) \cdot \int z_\mu(t)\, f(t)\, dt \tag{20}$$

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* We changed the symbol $x^{(\nu)}$ into ${}^\nu x$ from that used in our first paper (Bibl.1) so as to prevent confusion with the derivatives.

with the summation indices [see (Bibl.1), eqs.(66) and (70)]:

$$\begin{cases} (\nu) = m_1 + m_2 + \ldots + m_{\nu-1} + 1 , \\ [\nu] = m_1 + m_2 + \ldots + m_\nu . \end{cases} \tag{21}$$

For the components ${}^{\nu}x(t)$ with $\nu = \rho + 1, \ldots, s$, we can assume that they are functions periodic with P. These are uniquely determined [see (Bibl.1) theorem 16].

In the case $\nu = 1, 2, \ldots, \rho$, the following is obtained in accordance with our first paper [(Bibl.1) eqs.(109) and (106)]:

$${}^{\nu}x(t) = \sum_{\mu=0}^{m_\nu} \frac{t^{m_\nu-\mu}}{(m_\nu-\mu)!}\, {}^{\nu}\Theta_\mu(t) , \tag{22}$$

where ${}^{\nu}\Theta_\mu(t)$ has the form

$${}^{\nu}\Theta_\mu(t) = \sum_{\gamma=0}^{\rho} {}^{\nu}c_{\gamma\mu}\, q_{(\nu)+\gamma}(t) \quad \text{for } \mu = 0,1,\ldots,m_\nu - 1, \tag{23}$$

$${}^{\nu}\Theta_{m_\nu}(t) = \sum_{\gamma=0}^{\rho-1} \left({}^{\nu}c_{\gamma m_\nu} + v^{*}_{(\nu)+\gamma}(t) \right) q_{(\nu)+\gamma}(t) \tag{24}$$

The $v^{*}_{(\nu)+\gamma}(t)$ defined elsewhere [(Bibl.1) eq.(99)] are of no importance here. /71

It is further found that the constants ${}^{\nu}c_{\gamma\mu}$, aside from depending on ν, depend only on the difference $\mu - \gamma$; thus, using

$${}^{\nu}c_{\gamma\mu} = {}^{\nu}d_{\mu-\gamma} \tag{25}$$

eq.(23) can be replaced by

$${}^{\nu}\Theta_\mu(t) = \sum_{\gamma=0}^{\rho} {}^{\nu}d_{\mu-\gamma}\, q_{(\nu)+\gamma}(t) \text{ for } \mu = 0,1,\ldots,m_\nu - 1; \tag{26}$$

with the procedure being the same for eq.(24). The quantities ${}^{\nu}d_{\mu-\gamma}$ with $\mu - \gamma > 0$ are arbitrary integration constants, whereas

$${}^{\nu}d_0 = {}^{\nu}c_{\mu\mu} = \frac{1}{P}\, a_{[\nu]} \quad (\mu = 0,1,\ldots,m_\nu - 1) \tag{27}$$

with [see (Bibl.1), eqs.(86), (92), (93) and note $n^{z}_{[\nu]}(t) = z_{[\nu]}(t)$ according to eq.(10)]

$$a_{[\nu]} = \int_0^P \delta^{\tau}_{[\nu]}(\tau) f(\tau)\, d\tau = \int_0^P z_{[\nu]}(\tau) f(\tau)\, d\tau . \tag{28}$$

In the resonance subcase ($\nu = 1, \ldots, \sigma$) the quantity ${}^{\nu}x(t)$ will always be a polynomial of the degree m_ν with the coefficients ${}^{\nu}\Theta_\mu(t)$ which is periodic with P, while the coefficient ${}^{\nu}\Theta_0(t)$ of the highest power is obtained from eqs.(14), (26), and (27) as

$$ {}^{\nu}\Theta_0(t) = \frac{1}{P} a_{[\nu]} y_{(\omega)}(t) \quad (\nu = 1, \ldots, \sigma), \tag{29} $$

which means that it is an eigenfunction of the homogeneous differential equation (6) periodic with P. If, in the exceptional subcase, ${}^{\nu}d\beta_\nu$ with the smallest $\beta_\nu > 0$ is the first nonvanishing coefficient (25), we will obtain in an analogous manner, provided that $\beta_\nu < m_\nu$,

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$$ \left.\begin{array}{l} {}^{\nu}\Theta_\alpha(t) \equiv 0 \qquad \text{for } 0 \leq \alpha \leq \beta_\nu - 1 \\ {}^{\nu}\Theta_{\beta_\nu}(t) = {}^{\nu}d_{\beta_\nu} y_{(\omega)}(t) \end{array}\right\} \nu = \sigma+1, \ldots, s, \quad \beta_\nu < m_\nu . \tag{30} $$

In what follows, eq.(22), omitting the highest t-powers with vanishing coefficients, will be written in the following form:

$$ {}^{\nu}x(t) = \sum_{\mu=0}^{\omega_\nu} \frac{t^{\omega_\nu - \mu}}{(\omega_\nu - \mu)!} {}^{\nu}\Theta_\mu(t) \quad \text{with} \quad \begin{cases} \omega_\nu = m_\nu \text{ for } \nu = 1, \ldots, \sigma \\ \omega_\nu = m_\nu - \beta_\nu < m_\nu \text{ for } \nu = \sigma+1, \ldots, s . \end{cases} \tag{31} $$

A solution (19) of eq.(1) which is represented in the form (31) for $\nu = 1, \ldots, \rho$ and which is periodic with P for $\nu = \rho + 1, \ldots, s$, will be denoted here as the "normal solution". Such a normal solution can be written in the form of

$$ x(t) = \sum_{\delta=0}^{\omega} \psi_\delta(t) \cdot \frac{t^{\omega-\delta}}{(\omega-\delta)!} \tag{32} $$

where

$$ \omega = \max_{(\nu = 1, \ldots, s)} (\omega_\nu, 0) \tag{33} $$

is used and where the $\psi_\delta(\ )$ are composed of the ${}^{\nu}\Theta_\mu(t)$ in eqs.(26) and (24) to yield functions periodic with P.

It should be emphasized again that, in eq.(32), only one special solution of the principal subcases is used, namely, the solution periodic with P, whereas in the resonance and exceptional subcases the complete solution is used with m_ν arbitrary integration constants. /73

In view of eqs.(29) and (30), the following theorem can be confirmed:

Theorem 1: If, in eq.(33), $\omega > 0$, then the factor $\Psi_0(t)$ of the highest power t^ω occurring in eq.(32) will be a nonidentically vanishing solution of the homogeneous differential equation (6), periodic with P.

It is directly obvious:

Theorem 2: If the resonance subcase is valid for $\nu = 1, \ldots, \sigma > 0$, the order of the powers of x(t) in eq.(32) will be at least

$$\omega \geq \max_{(\nu=1,\ldots,\sigma)} \omega_\nu = \max_{(\nu=1,\ldots,\sigma)} m_\nu . \tag{34}$$

and at least one solution x(t) will exist for which

$$\omega = \max_{(\nu=1,\ldots,\sigma)} \omega_\nu .$$

This latter statement follows directly from a consideration of eqs.(26), (27), and (31).

A successive differentiation of eq.(32) to t yields

$$x^{(k)}(t) = \sum_{\delta=0}^{\omega} \left(\gamma_\delta^{(k)} t + \binom{k}{1} \gamma_{\delta-1}^{(k-1)} t + \binom{k}{2} \gamma_{\delta-2}^{(k-2)} t + \cdots + \gamma_{\delta-k} \right) \frac{t^{\omega-\delta}}{(\omega-\delta)!} , \tag{35}$$

$$k = 1,2,\ldots,(n-1),$$

where the formally written functions $\Psi_\beta^{(\alpha)}(t)$ with $\beta < 0$ must be replaced by zero. From this, the following is directly obtained:

Theorem 3: If $\Psi_0(t)$ in eq.(32) is not constant, all derivatives $x^{(k)}(t)$ of eq.(32), with $k = 1, \ldots, n-1$, have the same sequence of power increment t^ω as x(t) itself. If, conversely, $\Psi_0, \Psi_1, \ldots, \Psi_{\ell-1}$ are constant whereas $\Psi_\ell(t)$ is

not constant, the power orders of the derivatives of x(t) decrease successively by 1 down to $x^{(l)}(t)$ and, from then on, remain constant equal to $\omega - l$.

Section 2. The Case $a_n(t) \not\equiv 0$ /74

Considering that, in the case of

$$a_n(t) \not\equiv 0 \tag{36}$$

the homogeneous differential equation (6) in eq.(1) cannot have constant solutions, it follows directly from theorems 1 and 3:

Theorem 4: Under the condition (36), all derivatives $x^{(k)}(t)$ ($k = 0, 1, \ldots, n-1$) of a normal solution (32) of eq.(1) always have the same power order.

Note: The only exceptional case of theorem 4 can possibly be the case of

$$x(t) = \text{Const.} = C \neq 0 \tag{37}$$

which occurs for

$$f(t) = Ca_n(t) \tag{38}$$

since, in that case, $x^{(l)}(t) \equiv 0$ for $l \geq 1$. It is worthwhile to group this trivial but interesting special case with the general considerations.

Primarily, the resonance subcase must not occur here since then terms with t-powers would necessarily enter in eq.(32). Consequently, the assumption of a solution $_3z_\alpha(t)$ of eq.(11), periodic with P, for which [see eqs.(28) and (38)]

$$\int_0^P {}_4z_\alpha(\tau)\, a_n(\tau)\, d\tau = a_\alpha \neq 0 \tag{39}$$

is valid, must be continued up to contradiction. This contradiction is obtained directly from the first and last equations of the system (10), in view of eq.(39), since $_1z_\alpha(t)$, and thus also $_3z_\alpha(t)$, does not have the period P [see eq.(12)].

Consequently, for $\nu = 1, \ldots, \rho$ the exceptional case exists while for $\nu = \rho + 1, \ldots, s$ the principal case is present; here $\rho = 0$ and $\rho = s$ is admis- /75

sible.

First, let us calculate the vector

$$\mathfrak{b}_\nu(t) = \begin{bmatrix} b_{(\nu)}(t) \\ \vdots \\ b_{[\nu]}(t) \end{bmatrix} \tag{40}$$

for the special case involved here. According to another paper [(Bibl.1), eqs.(85) and (31)] and to the notations in that same paper and in eq.(76) (see footnote on p.32), we obtain

$$\mathfrak{b}(t) = \begin{bmatrix} \vdots \\ b_\nu(t) \\ \vdots \end{bmatrix} = \mathfrak{A}^T \mathfrak{x} = e^{\mathfrak{K}t} \mathfrak{Z}^T \mathfrak{x} = e^{\mathfrak{K}t} \begin{bmatrix} \vdots \\ z_{(\nu)}\mathfrak{x} \\ \vdots \\ z_{[\nu]}\mathfrak{x} \\ \vdots \end{bmatrix}.$$

Consequently, taking eq.(38) as well as the first and last equations of the system (10) into consideration, we have

$$\mathfrak{b}_\nu(t) = C \cdot e^{\mathfrak{K}_\nu t} \begin{bmatrix} {}_1z'_{(\nu)} \\ \vdots \\ {}_1z'_{[\nu]} \end{bmatrix} \quad \text{with} \quad e^{\mathfrak{K}_\nu t} = e^{\alpha_\nu t} \cdot \begin{bmatrix} 1 & t & \cdots & \frac{t^{m_\nu-1}}{(m_\nu-1)!} \\ & 1 & \cdots & \frac{t^{m_\nu-2}}{(m_\nu-2)!} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \tag{41}$$

in accordance with eq.(16).

We then define the row vectors

$${}_1\bar{\mathfrak{A}}_\nu^T = ({}_1\mathfrak{A}_{(\nu)}, \ldots, {}_1\mathfrak{A}_{[\nu]}) \quad \text{and} \quad {}_1\mathfrak{z}_\nu^T = ({}_1z_{(\nu)}, \ldots, {}_1z_{[\nu]}). \tag{42}$$

According to eq.(15),

$$\mathfrak{A}(t) = \mathfrak{Z}(t)\, e^{\mathfrak{K}_\nu^T t} = \left(e^{\mathfrak{K}_\nu t} \cdot \mathfrak{Z}_\nu^T(t)\right)^T; \tag{43}$$

so that the following is valid for the corresponding column vectors:

$${}_1\bar{\mathfrak{A}}_\nu(t) = (\bar{\mathfrak{A}}_\nu^T)^T = e^{\mathfrak{K}_\nu t}\, {}_1\mathfrak{z}_\nu(t). \tag{44}$$

Equation (74) (Bibl.1), namely,

$$\mathfrak{x}_\nu' = \mathfrak{K}_\nu \mathfrak{x}_\nu + \mathfrak{b}_\nu(t) \tag{45}$$

as readily verified by eqs.(44) and (41), has the solution

$$\varphi_\nu(t) = C\,\bar{\psi}_\nu(t), \qquad \nu = 1,2,\ldots,s. \tag{46}$$

periodic with P. Thus, eq.(32) according to [(Bibl.1), eqs.(79) and (77)] can be written in the following form:

$$x(t) = C\cdot\sum_{\nu=1}^{\Delta}\sum_{\beta=0}^{m_\nu-1}\varphi_{(\nu)+\beta}(t)\,\psi_{(\nu)+\beta}(t) = C\sum_{\lambda=1}^{n}\varphi_\lambda(t)\psi_\lambda(t) = C, \tag{47}$$

where, in the last transformation, $\Phi\Psi^T = \mathfrak{E}$ was used in accordance with eq.(15).

Section 3. The Differential Equation Reduced in Order /77

Since the case (36) in the previous Section was taken care of, we can assume below that*

$$a_n(t) \equiv a_{n-1}(t) \equiv \ldots \equiv a_{n-j+1}(t) \equiv 0,\ a_{n-j}(t) \not\equiv 0,$$
$$1 \leqslant j \leqslant n-1 \text{ 3)}. \tag{48}$$

Using

$$\hat{x}(t) = x^{(j)}(t), \quad \hat{y}(t) = y^{(j)}(t), \tag{49}$$

eqs.(1) and (6), respectively, are transformed into

$$\hat{L}[\hat{x}] \equiv \hat{x}^{(n-j)} + a_1(t)\,\hat{x}^{(n-j-1)} + \ldots + a_{n-j}(t)\hat{x} = f(t), \tag{50}$$

$$\hat{L}[\hat{y}] \equiv \hat{y}^{(n-j)} + a_1(t)\,\hat{y}^{(n-j-1)} + \ldots + a_{n-j}(t)\hat{y} = 0. \tag{51}$$

The pertaining adjoint homogeneous equation will then be

$$\bar{\hat{L}}[\hat{z}] \equiv (-1)^{n-j}\hat{z}^{(n-j)} + (-1)^{n-j-1}\left(a_1(t)\hat{z}\right)^{(n-j-1)} + \ldots + a_{n-j}(t)\hat{z} = 0. \tag{52}$$

Each solution of eq.(52) simultaneously is a solution of eq.(9).

Conversion of the differential equations (50), (51), and (52) into the corresponding differential equation systems of the first order, together with

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* The trivial case j = n leads to the differential equation $x^{(n)}(t) = f(t)$ with the reduced equation [see eq.(49) and (50)] $\hat{x} \equiv x^{(n)}(t) = f(t)$. The resonance case or the exceptional case is present depending on whether the mean value $\frac{1}{P}\int_0^P f(\tau)d\tau$ is not equal or is equal to zero.

$$\hat{c} = \begin{bmatrix} \hat{x} \\ \hat{x}' \\ \vdots \\ \hat{x}^{(n-\dot{j}-1)} \end{bmatrix}, \hat{y} = \begin{bmatrix} \hat{y} \\ \hat{y}' \\ \vdots \\ \hat{y}^{(n-\dot{j}-1)} \end{bmatrix}, \hat{j} = \begin{bmatrix} {}_1\hat{z} \\ {}_2\hat{z} \\ \vdots \\ {}_{n-\dot{j}}\hat{z} \end{bmatrix} \text{ and } \hat{f}(t) = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(t) \end{bmatrix} \tag{53}$$

will yield

$$\hat{c}' = \hat{A}(t)\,\hat{c} + \hat{f}(t)\,, \tag{54}$$

$$\hat{y}' = \hat{A}(t)\,\hat{y} \tag{55}$$

and

$$\hat{j}' = -\hat{A}^{T}(t)\,\hat{j} \tag{56}$$

with the matrix

$$\hat{A}(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_{n-\dot{j}}(t), & -a_{n-\dot{j}-1}(t), & \cdots, & -a_2(t), & -a_1(t) \end{bmatrix}, \tag{57}$$

which is correlated with the matrix $\mathfrak{A}(t)$ as follows [see (Bibl.1), eq.(23)]:

$$A(t) = \left[\begin{array}{ccc|c} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & \\ \hline & 0 & & & \hat{A}(t) \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{J}_0 & 0 \\ \hline 0 & \hat{A}(t) \end{array}\right] \tag{58}$$

Now let

$$\hat{\mathfrak{y}}(t) = \left(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{n-\dot{j}}\right) = \begin{bmatrix} \hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_{n-\dot{j}} \\ \hat{y}_1' & \hat{y}_2' & \cdots & \hat{y}_{n-\dot{j}}' \\ \vdots & & & \vdots \\ \hat{y}_1^{(n-\dot{j}-1)} & \hat{y}_2^{(n-\dot{j}-1)} & \cdots & \hat{y}_{n-\dot{j}}^{(n-\dot{j}-1)} \end{bmatrix} = \tag{59}$$

$$= \begin{bmatrix} y_1^{(\dot{j})} & y_2^{(\dot{j})} & \cdots & y_{n-\dot{j}}^{(\dot{j})} \\ y_1^{(\dot{j}+1)} & y_2^{(\dot{j}+1)} & \cdots & y_{n-\dot{j}}^{(\dot{j}+1)} \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-\dot{j}}^{(n-1)} \end{bmatrix}$$

be a fundamental solution matrix of the reduced homogeneous differential equation (51) so that, for $\hat{\mathfrak{y}}(t)$, the following representation [see eqs.(14) and (19)] applies:

$$\hat{\mathfrak{y}}(t) = \hat{\Phi}(t) \cdot e^{\hat{\mathfrak{R}} t} = \sum_{\nu=1}^{\hat{s}} {}^{\nu}\hat{\mathfrak{y}}(t) = \sum_{\nu=1}^{\hat{s}} {}^{\nu}\hat{\mathfrak{y}}(t) = \sum_{\nu=1}^{\hat{s}} {}^{\nu}\hat{\Phi}(t) e^{\hat{\mathfrak{R}}_{\nu} t} \tag{60}$$

with the fundamental subsystems:

$$ {}^{\nu}\hat{\mathfrak{y}}(t) = \left[\begin{array}{c|c|c} 0 & \begin{matrix} \hat{y}_{(\nu)} & \cdots & \hat{y}_{[\nu]} \\ \vdots & & \vdots \\ \hat{y}_{(\nu)}^{(n-j-1)} & \cdots & \hat{y}_{[\nu]}^{(n-j-1)} \end{matrix} & 0 \end{array} \right] = \tag{61}$$

$$ = \left[\begin{array}{c|c|c} 0 & \begin{matrix} y_{(\nu)}^{(i)} & \cdots & y_{[\nu]}^{(i)} \\ \vdots & & \vdots \\ y_{(\nu)}^{(n-1)} & & y_{[\nu]}^{(n-1)} \end{matrix} & 0 \end{array} \right]$$

and the periodic submatrices:

$$ {}^{\nu}\hat{\Phi}(t) = \left[\begin{array}{c|c|c} 0 & \begin{matrix} {}_{1}\hat{\varphi}_{(\nu)} & \cdots & {}_{1}\hat{\varphi}_{[\nu]} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ {}_{n-j}\hat{\varphi}_{(\nu)} & \cdots & {}_{n-j}\hat{\varphi}_{[\nu]} \end{matrix} & 0 \end{array} \right] . \tag{62}$$

Again, let us assume [see eq.(17)] that

$$\alpha_{\nu} \begin{cases} = 0 & \text{for } \nu = 1, \ldots, \hat{\rho}, \\ \neq 0 & \text{for } \nu = \hat{\rho}+1, \ldots, \hat{s}, \end{cases} \tag{63}$$

where, as always in special cases, $\hat{\rho}$ can be equal to zero or $\hat{s}$. In that case, the matrix $\hat{\mathfrak{N}}(t)$ can be expanded in the following manner to a fundamental solution matrix $\mathfrak{y}(t)$ of eq.(6) [see eq.(58)]:

$$\mathfrak{y}(t) = (y_1, \ldots, y_n) = \left[\begin{array}{c|c} \begin{matrix} 1 & t & \cdots & \frac{t^{j-1}}{(j-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{matrix} & \begin{matrix} y_1 & \cdots & y_{n-j} \\ \vdots & & \vdots \\ y_1^{(j-1)} & \cdots & y_{n-j}^{(j-1)} \end{matrix} \\ \hline 0 & \begin{matrix} y_1^{(j)} & \cdots & y_{n-j}^{(j)} \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_{n-j}^{(n-1)} \end{matrix} \end{array}\right] = \left[\begin{array}{c|c} e^{-\gamma_0 t} & \mathfrak{y}_0^* \\ \hline 0 & \hat{\mathfrak{y}}^* \end{array}\right] . \tag{64}$$

Here, the square matrix $\mathfrak{y}^*(t)$ is successively constructed by integrating the first row of the matrix $\hat{\mathfrak{y}}(t)$ in which case the integration constants can be arbitrarily selected.

Next, we partition the matrix $\mathfrak{y}^*(t)$, analogous to eq.(61), in the form of

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$$\mathfrak{y}^*(t) = \sum_{1}^{\hat{\rho}} {}^{\nu}\mathfrak{y}^*(t) \tag{65}$$

into the square submatrices ${}^{\nu}\mathfrak{y}^*$.

Then, we can formulate the following theorem.

Theorem 5: If the reduced differential equation (51) has exactly $\hat{\rho}$ independent solutions $\hat{y}_{(1)}, \hat{y}_{(2)}, \ldots, \hat{y}_{(\hat{\rho})}$ periodic with P, then the differential equation (6) either will have also exactly $\hat{\rho}$ independent solutions, periodic with P, or else eq.(6) will have exactly $\hat{\rho} + 1$ independent solutions periodic with P. In this case, the following is valid [see the notations of eq.(21)]:

1) If, for all $\hat{y}_{(\nu)}(t)$ ($\nu = 1, \ldots, \hat{\rho}$),

$$\int_0^P \hat{y}_{(\nu)}(t)dt = 0, \tag{66}$$

applies, then eq.(6) will have exactly $\hat{\rho} + 1$ independent solutions periodic with P.

2) If, conversely, for at least one $\hat{y}_{(\nu)}(t)$ ($\nu = 1, \ldots, \hat{\rho}$), the following mean value

$$\frac{1}{P}\int_0^P \hat{y}_{(\nu)}(t)dt = \hat{a}_\nu \neq 0 , \tag{67}$$

applies, then eq.(6) will have exactly $\hat{\rho}$ independent solutions periodic with P.

For a proof, the following simple auxiliary theorem is required:

Auxiliary theorem: Let g(t) be a function periodic with P, for which

$$\int_0^P g(t)dt = 0 \tag{68}$$

is valid. Then, exactly one function $h(t) = \int g(t)dt$ periodic with P will exist, for which

$$\int_0^P h(t)dt = 0 \tag{69}$$

is also valid.

Proof: For each constant h(o), /82

$$h(t) = \int_0^t g(\tau)\, d\tau + h(o)$$

is periodic with P. For a uniquely defined constant h(o), eq.(69) will then apply.

Application of the auxiliary theorem for proving the theorem 5 proceeds as follows:

1) Let $\int_0^P \hat{y}_{(\nu)}(t)dt = 0$ (for $\nu = 1,\ldots,\hat{\rho}$) [see eq.(66)].

By j integrations of the functions $\hat{y}_{(\nu)}(t)$ ($\nu = 1, \ldots, \hat{\rho}$), in which case the integration constants must be determined each time in accordance with the auxiliary theorem, exactly one function $y_{(\nu)}(t)$ having a mean value of zero and being periodic with P will be obtained for each $\hat{y}_{(\nu)}(t)$. Accordingly, based on the trivial solution $\hat{y}_0(t) \equiv 0$, the function $y_0(t) \equiv 1$ will be obtained as a further solution of eq.(6) periodic with P. That this solution is independent of the above-defined solutions $y_{(\nu)}(t)$ ($\nu = 1, 2, \ldots, \hat{\rho}$) follows from the fact that all these $y_{(\nu)}(t)$ have the mean value 0, while $y_0(t)$ has the mean value 1. That, in addition, the $y_{(\nu)}(t)$ ($\nu = 1, \ldots, \hat{\rho}$) are also mutually and linearly inde-

pendent follows from the linear independence of the functions $\hat{y}_{(\nu)}(t)$ after j differentiations.

2) Conversely, if not all $\hat{y}_{(\nu)}(\nu = 1, \ldots, \hat{\rho})$ have a mean value of zero, new solutions can be obtained by linear combination which, again, will be denoted by $\hat{y}_{(1)}, \ldots, \hat{y}_{(\hat{\rho})}$, that can be so defined that, for example, $\hat{y}_{(1)}$ has the mean value 1 while the remaining $y_{(\nu)}(\nu = 2, \ldots, \hat{\rho})$ have the mean values 0, i.e.,

$$\begin{cases} \frac{1}{P} \int_0^P \hat{y}_{(\nu)}(t)\,dt = 1 , \\ \int_0^P \hat{y}_{(\nu)}(t)\,dt = 0 \quad (\nu = 2, \ldots, \hat{\rho}) . \end{cases} \tag{70}$$

By j integrations of the functions $\hat{y}_{(\nu)}(\nu = 2, \ldots, \hat{\rho})$ and in accordance with /83 the auxiliary theorem, again $\hat{\rho} - 1$ solutions $y_{(\nu)}(t)$ of eq.(6), periodic with P and having a mean value of zero, will be obtained which are mutually and linearly independent. An integration of $\hat{y}_{(1)}(t)$ would yield a solution $y_{(1)}(t)$ not periodic with P. However, the trivial solution $\hat{y}_0(t) \equiv 0$ again leads to the solution $y_0(t) \equiv 1$ periodic with P which, together with the functions $y_{(2)}(t)$, $\ldots$, $y_{(\hat{\rho})}(t)$, forms a system of $\hat{\rho}$ linearly independent solutions of eq.(6), periodic with P.

Finally, it is easy to demonstrate that eq.(6) can have no further solution periodic with P. Let y(t) be any nonconstant solution of eq.(6) periodic with P; then, $\hat{y}(t) = y^{(j)}(t)$ will be a solution of the reduced differential equation (51) periodic with P which, in addition, has a mean value of 0 because of the differentiation process; consequently, $y^{(j)}(t)$ must be linearly composable of the already known solutions $\hat{y}_{(\nu)}(t)$ $(\nu = 1, \ldots, \hat{\rho})$.

It should also be mentioned that the solution vectors $\hat{\mathfrak{z}}(t)$ of eq.(56) must be completed into the solution vectors $\mathfrak{z}(t)$ of eq.(11) by a successive differ-

entiation process in accordance with eq.(10), toward components with smaller indices.

Section 4. Definition of a Special Normal Form of the Fundamental System of the Reduced Homogeneous Differential Equation

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Unfortunately, for solving the reduced system of differential equations (54), it is not sufficient to merely bring the matrix $\hat{R}$, introduced in accordance with eq.(14), to the Jordan normal form, in view of the fact that the main purpose of the theoretical consideration is a discussion of the system (4). For this, it is necessary to obtain the square matrix $\mathfrak{y}^*(t)$ occurring in eq.(64) in as concise and simple a form as possible. For this purpose, the fundamental system (59) of solutions of the reduced partial equation (51) is brought to a special normal form, which we will characterize by the following properties of the matrices (62) occurring in eq.(60):

Definition:

1) At $\lambda_\nu = 0$, either the following mean values are valid for $\mu = (\nu), \ldots, [\nu]$

$$\frac{1}{P}\int_0^P \hat{\varphi}_\mu(t)dt = 0 \quad \text{for all } \mu \tag{71}$$

or an index $0 \leq i_\nu \leq \hat{m}_\nu - 1$ ($\hat{m}_\nu = [\nu] - (\nu) + 1$) exists, so that the following is valid for the mean value:

$$\begin{cases} \frac{1}{P}\int_0^P \hat{\varphi}_{(\nu)+i_\nu}(t)\,dt = 1, \\ \frac{1}{P}\int_0^P \hat{\varphi}_{(\nu)+k}(t)\,dt = 0 \quad \text{for } k \neq i_\nu. \end{cases} \tag{72}$$

2) The elementary components are so arranged that, for $\nu = 1, \ldots, \lambda$, subscripts i_ν exist whereas, for $\nu = \lambda + 1, \ldots, \hat{\rho}$, all mean values (71) are equal to zero. For $\nu = \hat{\rho} + 1, \ldots, \hat{s}$, we have $\alpha_\nu = 0$; here, the elementary components are not restricted.

3) The orders $\hat{m}_\nu$ increase monotonically for $\nu = 1, \ldots, \lambda$:

$$\hat{m} > \hat{m}_{\nu-1}, \quad \nu = 1, \ldots, \lambda, \tag{73}$$

so that the inequalities /85

$$i_\nu > i_{\nu-1}, \quad \hat{m}_\nu - i_\nu > \hat{m}_{\nu-1} - i_{\nu-1} \quad \text{for } \nu = 2, \ldots, \lambda. \tag{74}$$

are valid. The elementary components for which all mean values vanish are also arranged in accordance with increasing values of $\hat{m}_\nu$ except that, in this case, only

$$\hat{m} \geqq \hat{m}_{\nu-1}, \quad \nu = \lambda + 2, \ldots, \hat{\rho}. \tag{75}$$

applies.

For constructing this special normal form, we will need several new concepts and theorems.

Let [see eq.(14) and the notation (42)], under the assumption of $\alpha_\nu = 0$,

$$\hat{\vartheta}_\nu^T = \left(\hat{\jmath}_{(\nu)}, \ldots, \hat{\jmath}_{[\nu]}\right) = \left(\hat{\varphi}_{(\nu)}, \ldots, \hat{\varphi}_{[\nu]}\right) \begin{bmatrix} 1 & t & \cdots & \frac{t^{\hat{m}_\nu - 2}}{(\hat{m}_\nu - 2)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{bmatrix} = \tag{76}$$

$$= \left(\hat{\varphi}_{(\nu)}, t\hat{\varphi}_{(\nu)} + \hat{\varphi}_{(\nu)+1}, \ldots, \frac{t^{\hat{m}_\nu - 1}}{(\hat{m}_\nu - 1)!}\hat{\varphi}_{(\nu)} + \cdots + \hat{\varphi}_{[\nu]}\right)$$

be a fundamental subsystem belonging to the ν^{th} elementary component. Either all mean values are

$$M_\mu = \frac{1}{P}\int_0^P \hat{\varphi}_\mu(t)dt = 0 \quad \text{for } \mu = (\nu), \ldots, [\nu], \tag{77}$$

or one i_ν exists, so that

$$M_\mu \begin{cases} = 0 & \text{for} \quad (\nu) \leqq \mu < (\nu) + i_\nu, \\ \neq 0 & \text{for} \quad \mu = (\nu) + i_\nu, \ 0 \leqq i_\nu \leqq \hat{m}_\nu - 1, \\ \text{arbitrarily for} & \quad \mu > (\nu) + i_\nu. \end{cases} \tag{78}$$

Then, the following applies:

Theorem 6: A regular matrix

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$$ {}^{\nu}\mathfrak{L} = \begin{bmatrix} c_{(\nu)} & c_{(\nu)+1} & \cdots & c_{[\nu]} \\ & \ddots & \ddots & \vdots \\ & & \ddots & c_{(\nu)+1} \\ & & & c_{(\nu)} \end{bmatrix} = c_{(\nu)}\hat{\mathfrak{E}}_{\nu} + c_{(\nu)+1}\hat{\mathfrak{J}}_{\nu} + \cdots + c_{[\nu]}\hat{\mathfrak{J}}_{\nu}^{\hat{n}_{\nu}-1} \tag{79} $$

is in existence [see the denotation (58)], so that the transformed fundamental system ${}_1\hat{\mathfrak{h}}_{\nu}^{T} \cdot {}^{\nu}\mathfrak{C}$ has the property 1.

Proof: Since the matrix ${}^{\nu}\mathfrak{C}$ is commutative with the matrix

$$ e^{\hat{\mathfrak{J}}t} = \begin{bmatrix} 1 & t & \cdots & \frac{t^{\hat{n}_{\nu}-1}}{(\hat{n}_{\nu}-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{bmatrix} = \tag{80} $$

$$ = \hat{\mathfrak{E}}_{\nu} + t\hat{\mathfrak{J}}_{\nu} + \frac{t^2}{2}\hat{\mathfrak{J}}_{\nu}^{2} + \cdots + \frac{t^{\hat{n}_{\nu}-1}}{(\hat{n}_{\nu}-1)!}\hat{\mathfrak{J}}_{\nu}^{\hat{n}_{\nu}-1} $$

it is immaterial, because of eq.(14), whether the transformation ${}^{\nu}\mathfrak{C}$ is applied to the $\hat{y}_{\mu}$ or directly to the $\hat{\varphi}_{\mu}$. The row vector

$$ {}_1\bar{\hat{\varphi}}_{\nu}^{T} = \left(\hat{\varphi}_{(\nu)}, \cdots, \hat{\varphi}_{[\nu]}\right) \tag{81} $$

is then transformed into the column vector

$$ {}_1\bar{\hat{\varphi}}_{\nu}^{\circ T} = {}_1\bar{\hat{\varphi}}_{\nu}^{T} \cdot {}^{\nu}\mathfrak{L}^{-} . \tag{82} $$

If, for each subscript μ, the components

$$ \hat{\varphi}_{\mu}(t) = \tilde{\hat{\varphi}}_{\mu}(t) + M_{\mu} \tag{83} $$

are separated into the constant mean value M_{μ} [see eq.(78)] and into the function $\tilde{\hat{\varphi}}_{\mu}(t)$ of a mean value zero, the following applies in a readily understandable notation:

$$ {}_1\bar{\hat{\varphi}}_{\nu}^{\circ T} = {}_1\bar{\tilde{\hat{\varphi}}}_{\nu}^{T} \cdot {}^{\nu}\mathfrak{L}^{-} + \mathfrak{M}_{\nu}^{T} \cdot {}^{\nu}\mathfrak{L}^{-} . \tag{84} $$

The row vector $\mathfrak{M}_{\nu}^{T}$ is then to be transformed by the last summand in eq.(82) into the row vector $\mathfrak{M}_{\nu}^{\circ T}$, which is defined by /87

$$M^{\bullet}_{(\nu)+i_\nu} = 1 \;,\; M^{\bullet}_{\mu} = 0 \quad \text{for } \mu \neq (\nu)+i_\nu \tag{85}$$

This[see eq.(79)] leads to the system of equations:

$$\left\{\begin{array}{l} 0 = c_{(\nu)}\, M_{(\nu)} \\ 0 = c_{(\nu)}\, M_{(\nu)+1} + c_{(\nu)+1}\, M_{(\nu)} \\ \cdot \\ \cdot \\ \cdot \\ 0 = c_{(\nu)}\, M_{(\nu)+i_\nu-1} + \cdots + c_{(\nu)+i_\nu-1}\, M_{(\nu)} \\ 1 = c_{(\nu)}\, M_{(\nu)+i_\nu} + \cdots + c_{(\nu)+i_\nu}\, M_{(\nu)} \\ 0 = c_{(\nu)}\, M_{(\nu)+i_\nu+1} + \cdots + c_{(\nu)+i_\nu+1}\, M_{(\nu)} \\ \cdot \\ \cdot \\ \cdot \\ 0 = c_{(\nu)}\, M_{[\nu]} + \cdots + c_{(\nu)} M_{(\nu)} \end{array}\right. \tag{86}$$

The first i_ν equations are automatically satisfied because of eq.(78). The remaining equations successively lead to solutions: $c_{(\nu)}$, $c_{(\nu)+1}$, ..., $c_{[\nu]-i_\nu}$, in which case we definitely will have $c_{(\nu)} \neq 0$. This means that ${}^{\nu}\mathfrak{S}$ is regular. [In the case that, instead of eq.(62), all $M_\mu = 0$ (see eq.(77)) one can simply pose ${}^{\nu}\mathfrak{S} = \mathfrak{S}_\nu$.]

From now on we can assume that the fundamental subsystems ($\hat{y}_{(\nu)}$, ..., $\hat{y}_{[\nu]}$), for all ν, have at least the first property of the special normal form. The further properties can be established by means of the following theorem: /88

Theorem 7: Let, in addition to the fundamental subsystem (76), another fundamental subsystem be given:

$$\hat{\mathfrak{y}}_k^{\,T} = \left(\hat{\varphi}_{(k)},\; t\,\hat{\varphi}_{(k)} + \hat{\varphi}_{(k)+1},\; \ldots,\; \frac{t^{\hat{m}_k-1}}{(\hat{m}_k-1)!}\,\hat{\varphi}_{(k)} + \cdots + \hat{\varphi}_{[k]}\right), \quad (k \neq \nu), \tag{87}$$

so that, in contrast to eq.(74),

$$m_k - i_k \geq \hat{m}_\nu - i_\nu \tag{88}$$

the following simultaneously applies:

$$0 \leqq i_\nu - i_k < \hat{m}_\nu \tag{89}$$

The right-hand inequality in eq.(89) is trivially satisfied. Then, by superposition of these two systems, a fundamental subsystem ${}_1\hat{n}_\nu^{*\dagger}$ analogous to eq.(76) can be formed for which the functions $\hat{\Phi}_{(\nu)}^*, \ldots, \hat{\Phi}_{[\nu]}^*$, which are periodic with P all will have the mean value zero. In this case, the fundamental system ${}_1\hat{n}_k^{\dagger}$ remains unchanged.

Proof: If the new solutions $\hat{y}_\mu^*$ of eq.(51) are formed in accordance with the stipulation

$$\hat{y}_\mu^* = \begin{cases} y_\mu & \text{for } \mu = (\nu), (\nu)+1, \ldots, (\nu)+i_\nu - i_k - 1 \\ \hat{y}_\mu - \hat{y}_{(k)+\mu-((\nu)+i_\nu-i_k)} & \text{for } \mu = (\nu)+i_\nu-i_k, (\nu)+i_\nu-i_k+1, \ldots, (\nu)+\hat{m}_\nu-1=[\nu] \end{cases} \tag{90}$$

the following will be obtained, because of eq.(14), for the corresponding functions $\hat{\varphi}_\mu^*$:

$$\hat{\varphi}_\mu^* = \begin{cases} \hat{\varphi}_\mu & \text{for } \mu = (\nu), (\nu)+1, \ldots, (\nu)+i_\nu-i_k-1 \\ \hat{\varphi}_\mu - \hat{\varphi}_{(k)+\mu-((\nu)+i_\nu-i_k)} & \text{for } \mu = (\nu)+i_\nu-i_k, (\nu)+i_\nu-i_k+1, \ldots, (\nu)+\hat{m}_\nu-1=[\nu] \end{cases} \tag{91}$$

Here, eq.(89) guarantees the correct values of μ for eqs.(90) and (91) while eq.(88) ensures that

$$(k) \leqq (k)+\mu-((\nu)+i_\nu-i_k) \leqq [k]$$

is applicable. It can be confirmed that the functions $\hat{\varphi}_\mu^*$ from eq.(91) have /89 a zero mean value since, for $\mu = (\nu) + i_\nu$, we simultaneously have $(k) + \mu - ((\nu) + i_\nu - i_k) = (k) + i_k$. Thus, theorem 7 is proved.

Now, we can establish a certain normal form by also satisfying the conditions 2 and 3 of its definition. Each elementary component will be denoted by a pair of numbers $(\hat{m}_\nu, i_\nu)$ or $(\hat{m}_\nu, \hat{m}_\nu)$ if no i_ν exists. In addition, we calculate the differences $\hat{m}_\nu - i_\nu$ and define a sequence of number triples $(\hat{m}_\nu, i_\nu, \hat{m}_\nu - i_\nu)$ or $(\hat{m}_\nu, \hat{m}_\nu, 0)$. First, the subscripts ν with the smallest i_ν will be defined. Among these subscripts, one is selected for which $\hat{m}_\nu - i_\nu$ is as large

as possible. After this, the elementary components or the number triples ($\hat{m}_\nu$, i_ν, $\hat{m}_\nu - i_\nu$) are so rearranged that the above characterized number triple will be in the first position, after which we set the corresponding $\nu = 1$. According to theorem 7, all other elementary components for which eqs.(88) and (89) with $k = 1$ are satisfied, can be so transformed by superposition with the first elementary component that no more i_ν will exist for them. The already determined first elementary component will remain unchanged. Since, in any case $i_\nu \geq i_1$ and since, for $i_\nu = i_1$, we automatically have $\hat{m}_\nu - i_\nu \leq \hat{m}_1 - i_1$, this means that the possibly remaining number triples for which an i_ν still exists, will satisfy the conditions $i_\nu > i_1$ and $\hat{m}_\nu - i_\nu > \hat{m}_1 - i_1$. Of these remaining triples we again select one for which we will pose $\nu = 2$ so that, at minimal i_ν, the $\hat{m}_\nu - i_\nu$ becomes maximal. Here again, a superposition in accordance with theorem 7 will make it possible to transform the elementary components for which eqs.(88) and (89) with $k = 2$ are valid, together with the above-defined second elementary component, in such a manner that no i_ν will exist for them whereas the number triples for which an i_ν might still exist will satisfy the conditions

$$i_\nu > i_2 > i_1 \text{ and } \hat{m}_\nu - i_\nu > \hat{m}_2 - i_2 > \hat{m}_1 - i_1$$

We continue in this manner and finally obtain a sequence of elementary components with number triples ($\hat{m}_\nu$, i_ν, $\hat{m}_\nu - i_\nu$) for $\nu = 1, 2, \ldots, \lambda$, where the /90 following is valid for $\nu = 2, \ldots, \lambda$ (naturally, $\lambda = 0$ or $\lambda = 1$ is also possible):

$$i_\nu > i_{\nu-1} \cdot \hat{m}_\nu - i_\nu > \hat{m}_{\nu-1} - i_{\nu-1}$$

from which also $\hat{m}_\nu > \hat{m}_{\nu-1}$ follows directly. Then, for $\alpha_\nu = 0$ the elementary components with the characterizing triples ($\hat{m}_\nu$, $\hat{m}_\nu$, 0) for $\nu = \lambda + 1, \ldots, \hat{\rho}$ might possibly be left over, which can be so arranged that $\hat{m}_\nu \geq \hat{m}_{\nu-1}$ (for $\nu = \lambda + 2, \ldots, \hat{\rho}$) is valid. The elementary components with $\alpha_\nu \neq 0$ can remain

unchanged. This finally establishes the special normal form.

Section 5. Completion of the Special Normal Form of the Fundamental System of the Reduced Homogeneous Differential Equation into a Fundamental System of the Original Homogeneous Differential Equation

The purpose of this particular Section is to obtain proof of the following:

Theorem 8: Based on the special form (59) resp. (60) of the fundamental system of solutions of the differential equation (51), a fundamental system

$$\mathfrak{y}(t) = \Phi(t) \cdot \mathfrak{z}(t) \tag{92}$$

of solutions of the differential equation (6) can be obtained, in which case

$$\mathfrak{z}(t) = e^{\mathfrak{A}t}, \quad \Phi(t+P) = \Phi(t) \tag{93}$$

but where the constant matrix $\mathfrak{A}$ does not necessarily have the Jordan normal form [see also eqs.(108) and (114)].

The proof must proceed in several steps. Primarily, we establish:

Theorem 9: If, in the case $\alpha_\nu = 0$, eq.(71) is valid for all values of $\mu = (\nu), \ldots, [\nu]$ as well as in the case $\alpha_\nu \neq 0$*, the solution submatrix ${}^{\nu}\hat{\mathfrak{y}}$ [see eq.(61)] of eq.(51) can be thus completed by a matrix ${}^{\nu}\mathfrak{m}^*$ into a solution submatrix ${}^{\nu}\mathfrak{y}$ of eq.(6) such that, once more,

$${}^{\nu}\mathfrak{y} = \begin{bmatrix} {}^{\nu}\mathfrak{y}^* \\ {}^{\nu}\hat{\mathfrak{y}} \end{bmatrix} = e^{\alpha_\nu t} \begin{bmatrix} {}^{\nu}\Phi^* \\ {}^{\nu}\hat{\Phi} \end{bmatrix} e^{\hat{\mathfrak{A}}_\nu t} = {}^{\nu}\Phi(t) \cdot e^{\hat{\mathfrak{A}}_\nu t} \tag{94}$$

is valid.

Proof. First, let us consider the case $j = 1$. So that the direct argument

$$\mathfrak{y}_\nu' = e^{\alpha_\nu t}\, \mathfrak{z}_\nu^{\tau} \cdot e^{\hat{\mathfrak{A}}_\nu t} \tag{95}$$

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* And thus also $\alpha_\nu \neq \frac{2k\pi i}{P}$ [see (Bibl.1), eq.(20) with $q = P$].

[see eqs.(81) and (80)], after one differentiation, will lead to /92

$$ {}_1\hat{\mathfrak{y}}_\nu^{\mathsf{T}} = e^{\alpha_\nu t}\, {}_1\bar{\varphi}_\nu^{\mathsf{T}}\, e^{\hat{\mathfrak{F}}_\nu t} \tag{96} $$

the existence of the following differential equations for $\varphi_{(\nu)}(t)$, $\varphi_{(\nu)+1}(t)$, ..., $\varphi_{[\nu]}(t)$ will be necessary and sufficient:

$$ \begin{cases} \varphi'_{(\nu)} + \alpha_\nu \varphi_{(\nu)} = \hat{\varphi}_{(\nu)} \\ \varphi'_{(\nu)+1} + \alpha_\nu \varphi_{(\nu)+1} = \hat{\varphi}_{(\nu)+1} - \varphi_{(\nu)} \\ \vdots \\ \varphi'_{[\nu]} + \alpha_\nu \varphi_{[\nu]} = \hat{\varphi}_{[\nu]} - \varphi_{[\nu]-1}\,, \end{cases} \tag{97} $$

from which, because of the two possibilities of theorem 9, functions $\varphi_{(\nu)}(t)$, ..., $\varphi_{[\nu]}(t)$ periodic with P can be successively determined for α_ν; these will be defined uniquely in the case $\alpha_\nu \neq 0$ whereas, in the case $\alpha_\nu = 0$ if eq.(71) applies, they will be defined uniquely only if it is additionally stipulated that all mean values are zero. In the case of $j = 2$, the same method of reasoning is to be applied first to

$$ {}_1\mathfrak{y}'^{\mathsf{T}}_\nu = e^{\alpha_\nu t}\, {}_2\bar{\varphi}_\nu^{\mathsf{T}}\, e^{\hat{\mathfrak{F}}_\nu t} \tag{98} $$

instead of to eq.(95), after which the ${}_1\bar{\varphi}_\nu^{\mathsf{T}}$ is determined from eq.(95) according to the same syllogism, by substituting eq.(96) with eq.(98). The procedure is wholly similar for the remaining values of j. From this, eq.(94) of theorem 9 will follow if

$$ \hat{\mathfrak{K}}_\nu = \alpha_\nu \hat{\mathfrak{k}}_\nu + \hat{\mathfrak{F}}_\nu \tag{99} $$

is also taken into consideration [see the equation corresponding to the second relation (41)].

This leaves the case $\alpha_\nu = 0$, at validity of eq.(72), to be considered. Under introduction of the vectors n_ν^{T} with the components /93

$$ e_{(\nu)+\mu} = \begin{cases} 1 & \text{for } \mu = i_\nu \\ 0 & \text{for } \mu \neq i_\nu \end{cases} \tag{100} $$

we resolve

$$ {}_1\tilde{\varphi}_\nu^T = {}_1\bar{\varphi}_\nu^T + n_\nu^T . \tag{101} $$

As in the proof for theorem 9, we can first determine a uniquely defined row vector ${}_1\tilde{\varphi}_\nu^T$ with the mean value 0 so that the j^{th} derivative (note $\hat{\vartheta}_\nu = \hat{\Pi}_\nu$) of ${}_1\tilde{\varphi}_\nu^T \cdot e^{\hat{\vartheta}_\nu t}$ is exactly ${}_1\hat{\varphi}_\nu^T \cdot e^{\hat{\vartheta}_\nu t}$. By j integrations of $n_\nu^T \cdot e^{\hat{\vartheta}_\nu t}$ we obtain the row vector

$$ \left(0,\ldots,0,\frac{t^j}{j!},\frac{t^{j+1}}{(j+1)!},\ldots,\frac{t^{j+\hat{m}_\nu-i_\nu-1}}{(j+\hat{m}_\nu-i_\nu-1)!}\right) . \tag{102} $$

From this, we have:

Theorem 10: In the case $\alpha_\nu = 0$ and eq.(72), eq.(94) will be replaced by

$$ {}^\nu\eta = \begin{bmatrix} {}^\nu\eta^* \\ {}^\nu\hat{\eta} \end{bmatrix} \quad \text{with} \quad {}^\nu\eta^* = {}^\nu\tilde{\Phi}^* e^{\hat{\vartheta}_\nu t} + {}^\nu\varrho^*(t) \tag{103} $$

with

$$ {}^\nu\varrho^*(t) = \begin{bmatrix} 0,\ldots,0,\frac{t^j}{j!},\frac{t^{j+1}}{(j+1)!},\ldots,\frac{t^{j+\hat{m}_\nu-i_\nu-1}}{(j+\hat{m}_\nu-i_\nu-1)!} \\ \vdots \\ 0,\ldots,0,t,\frac{t^2}{2!},\ldots,\frac{t^{\hat{m}_\nu-i_\nu}}{(\hat{m}_\nu-i_\nu)!} \end{bmatrix} . \tag{104} $$

Instead of eq.(103), we can also write

$$ {}^\nu\eta = \begin{bmatrix} {}^\nu\tilde{\Phi}^* \\ {}^\nu\hat{\Phi} \end{bmatrix} e^{\hat{\vartheta}_\nu t} + \begin{bmatrix} {}^\nu\varrho^* \\ 0 \end{bmatrix} . \tag{105} $$

Finally, j integrations of the trivial solution system /94

$$ {}_1\hat{\vartheta}_0^T = (0,\ldots,0) \quad (j \text{ zeros}) $$

will yield the row vector

$$ {}_1\vartheta_0^T = \left(1,t,\ldots,\frac{t^{j-1}}{(j-1)!}\right) \tag{106} $$

from which the solution submatrix

$$ {}^\bullet\eta = \begin{bmatrix} {}^\bullet\varrho(t) \\ \hline 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \cdots & \frac{t^{j-1}}{(j-1)!} \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & t \\ & & & 1 \\ \hline 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} e^{\hat{\vartheta}_\bullet t} \\ \hline 0 \end{bmatrix} = \begin{bmatrix} e^{\hat{\vartheta} t} \\ \hline 0 \end{bmatrix} \tag{107} $$

is obtained. By adding the matrices ${}^{\nu}\Phi^{*}$ of ${}^{\nu}\tilde{\Phi}^{*}$ of theorems 9 and 10, we form the matrix $\Phi^{*}(t)$ and write

$$\Phi(t) = \begin{pmatrix} 1 & & & \\ & \ddots & & \Phi^{*} \\ & & 1 & \\ 0 & & & \hat{\Phi} \end{pmatrix} \tag{108}$$

Similarly, we form (see theorem 10)* the quantity $\mathfrak{H}^{*}(t)$ by addition of the matrices ${}^{\nu}\mathfrak{H}^{*}(t)$ according to eq.(104) and then, taking eq.(107) into consideration, put

$$\mathfrak{H}(t) = \begin{pmatrix} {}^{\circ}\mathfrak{H} & \mathfrak{H}^{*} \\ 0 & e^{\hat{\alpha}t} \end{pmatrix} \tag{109}$$

This will yield a fundamental solution matrix $\mathfrak{M}(t)$ of eq.(6) in the form of eq.(92), where already the second equation of the system (93) possesses validity. /95

To furnish a complete proof for theorem 8, it merely remains to be demonstrated that eq.(109) can also be written in the form of the first equation of the system (93). This is automatically the case if no $\alpha_{\nu} = 0$ with the validity of eq.(72) are present, i.e., if the theorem 10 need not be used. Then, the matrix $\mathfrak{H}(t)$ of eq.(109) will already have the conventional form:

$$\mathfrak{H}(t) = \begin{pmatrix} 1 & t & \cdots & \frac{t^{\delta-1}}{(\delta-1)!} & \\ & \ddots & \ddots & \vdots & 0 \\ & & \ddots & t & \\ & & & 1 & \\ 0 & & & & e^{\hat{\alpha}t} \end{pmatrix} = e^{\mathfrak{A}t} \tag{110}$$

with

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_{0} & 0 \\ 0 & \hat{\mathfrak{A}} \end{pmatrix} \tag{111}$$

* In the case of theorem 9, we must use ${}^{\nu}\mathfrak{H}^{*}(t) = 0$.

and

$$\mathcal{K}_0 = \mathcal{N}_0 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \tag{112}$$

of the order $m_0 = j$. Here, $\mathcal{R}$ has the Jordan normal form.

Generally, the following theorem applies: /96

Theorem 11: The matrix $\mathcal{Q}(t)$ according to eq.(109) has the form of

$$\mathcal{Q}(t) = e^{\mathcal{K} t} \tag{113}$$

with

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_0 & {}^{\bullet}\mathcal{K}_1 & \cdots & {}^{\bullet}\mathcal{K}_\lambda & 0 & \cdots & 0 \\ & \hat{\mathcal{K}}_1 & & & & & \\ & & \ddots & & & & \\ & & & \hat{\mathcal{K}}_\lambda & & & \\ & & & & \hat{\mathcal{K}}_{\lambda+1} & & \\ & & & & & \ddots & \\ & & & & & & \hat{\mathcal{K}}_s \end{bmatrix} \tag{114}$$

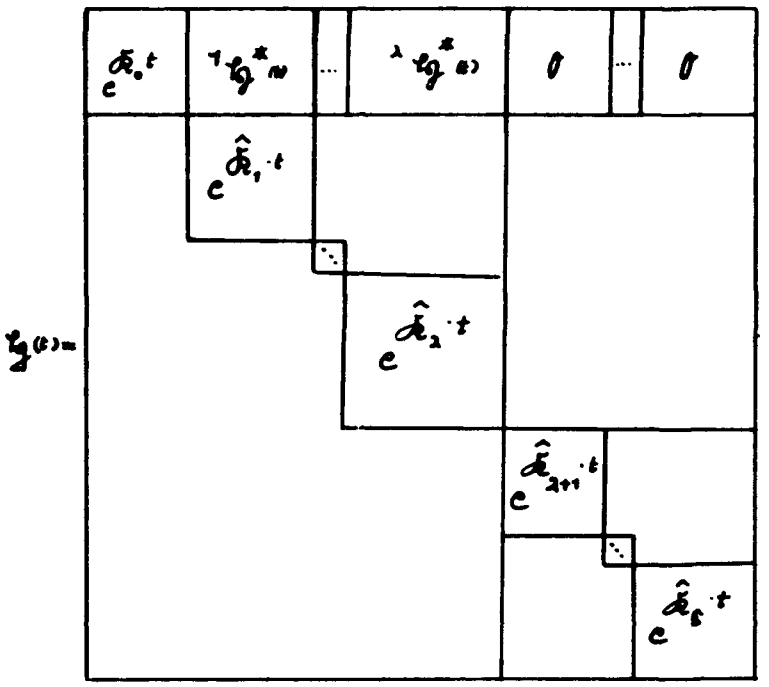
where, in addition to the components $\mathcal{R}_0, \hat{\mathcal{R}}_1, \ldots, \hat{\mathcal{R}}_s$, only the matrices* /97

$${}^{\bullet}\mathcal{K}_\nu = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 \cdots 0 & 1 & 0 \cdots & 0 \end{bmatrix}, \quad \nu = 1, \ldots, \lambda \tag{115}$$

* For the meaning of λ, see the definition given in Section 4. In the case $\lambda = 0$, eq.(116) is transformed into eq.(110) or eq.(114) into eq.(111).

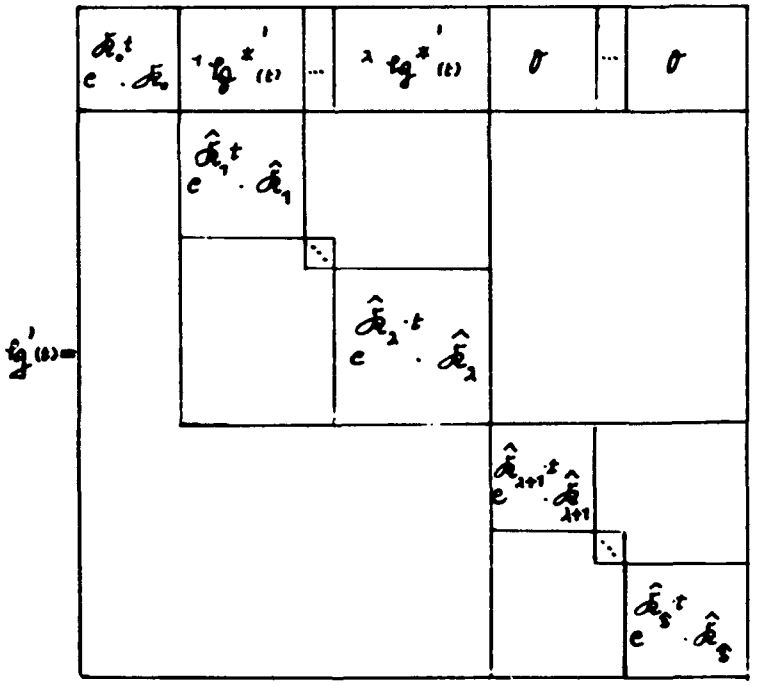
with a 1 in the j^{th} row occupy the position $1 + i_\nu$. In the case $\lambda > 0$, eq.(114) will not have the Jordan normal form.

Proof: Explicitly, the matrix $\mathfrak{H}(t)$ reads as follows [see eqs.(109), (104), (107)]:



(116)

By differentiation of $\mathfrak{H}(t)$ to t, we obtain



(117)

with (note: $\hat{\pi}_\nu = \hat{\vartheta}_\nu$ for $\nu = 1, \ldots, \lambda$)

$$
{}^{\nu}\mathfrak{y}^{*\prime}(\omega) = \begin{bmatrix} 0, \ldots, 0, \frac{t^{j-1}}{(j-1)!}, \frac{t^{j}}{j!}, \ldots, \frac{t^{j+\hat{\alpha}_\nu-\hat{\varkappa}_\nu-2}}{(j+\hat{\alpha}_\nu-\hat{\varkappa}_\nu-2)!} \\ \vdots \\ 0, \ldots, 0, 1, t, \ldots, \frac{t^{\hat{\alpha}_\nu-\hat{\varkappa}_\nu-1}}{(\hat{\alpha}_\nu-\hat{\varkappa}_\nu-1)!} \end{bmatrix} =
$$

$$
= \begin{bmatrix} 0, \ldots, 0, \frac{t^{j}}{j!}, \frac{t^{j+1}}{(j+1)!}, \ldots, \frac{t^{j+\hat{\alpha}_\nu-\hat{\varkappa}_\nu-1}}{(j+\hat{\alpha}_\nu-\hat{\varkappa}_\nu-1)!} \\ \vdots \\ 0, \ldots, 0, t, \frac{t^2}{2!}, \ldots, \frac{t^{\hat{\alpha}_\nu-\hat{\varkappa}_\nu}}{(\hat{\alpha}_\nu-\hat{\varkappa}_\nu)!} \end{bmatrix} \hat{\mathfrak{k}}_\nu + \begin{bmatrix} 0, \ldots, 0, \frac{t^{j-1}}{(j-1)!}, 0, \ldots, 0 \\ \vdots \\ 0, \ldots, 0, 1, 0, \ldots, 0 \end{bmatrix},
$$

i.e., according to eqs.(104), (107), and (115): /99

$$
{}^{\nu}\mathfrak{y}^{*\prime}(t) = {}^{\nu}\mathfrak{y}^{*}(t)\ \hat{\mathfrak{k}}_\nu + {}^{0}\mathfrak{y}(t)\cdot {}^{0}\mathfrak{k}_\nu \,. \tag{118}
$$

Hence, eq.(117) will yield

$$
\mathfrak{y}'(t) = \begin{bmatrix} e^{\mathfrak{k}_0 t}\cdot \mathfrak{k}_0 & {}^{1}\mathfrak{y}^{*}\hat{\mathfrak{k}}_1 + {}^{0}\mathfrak{y}\,{}^{0}\mathfrak{k}_1 & \cdots & {}^{\lambda}\mathfrak{y}^{*}\hat{\mathfrak{k}}_\lambda + {}^{0}\mathfrak{y}\,{}^{0}\mathfrak{k}_\lambda & 0 & \cdots & 0 \\ & e^{\hat{\mathfrak{k}}_1 t}\cdot \hat{\mathfrak{k}}_1 & & & & & \\ & & \ddots & & & & \\ & & & e^{\hat{\mathfrak{k}}_\lambda t}\cdot \hat{\mathfrak{k}}_\lambda & & & \\ & & & & e^{\hat{\mathfrak{k}}_{\lambda+1} t}\cdot \hat{\mathfrak{k}}_{\lambda+1} & & \\ & & & & & \ddots & \\ & & & & & & e^{\hat{\mathfrak{k}}_s t}\cdot \hat{\mathfrak{k}}_s \end{bmatrix} \tag{119}
$$

From this, on the basis of eqs.(116) and (114), the following equation can be read off:

$$
\mathfrak{y}'(t) = \mathfrak{y}(t)\cdot \mathfrak{k} \tag{120}
$$

Further, in accordance with eqs.(116) and (104), we obviously have

$$\mathfrak{y}_{(0)} = \mathfrak{E} . \tag{121}$$

However, eq.(113) follows from eqs.(120) and (121) [see, for example (Bibl.3), Sect.3.4].

As a corollary, we should note: For the matrix $\mathfrak{P}$, defined in another report [(Bibl.1), eq.(10)] and which is constant in accordance with the same paper (theorem 1), the following is valid: /100

$$\mathfrak{P} = \mathfrak{y}^{-1}(0)\,\mathfrak{y}(P) = \mathfrak{y}(P) = e^{\mathfrak{K}P} . \tag{122}$$

For this, besides eqs.(92) and (121), only the second equation in the system (93) as well as eq.(113) are needed.

Section 6. Construction of the Jordan Normal Form for $\mathfrak{K}$ /101

Now, we will have to bring the matrix $\mathfrak{K}$, defined in eq.(114), to the Jordan normal form $\mathfrak{K}^0$ by a similarity transformation:

$$\mathfrak{K}^0 = \mathfrak{L}^{-1}\,\mathfrak{K}\,\mathfrak{L} . \tag{123}$$

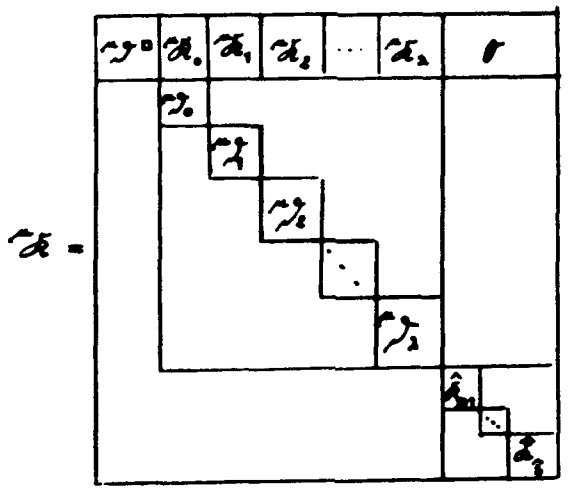
The pertinent fundamental solution matrix of eq.(6), according to eq.(14) and according to our first paper [(Bibl.1), eq.(14)] will then read

$$\mathfrak{y}^0(t) = \mathfrak{y}(t)\cdot\mathfrak{L} = \Phi^0(t)\cdot e^{\mathfrak{K}^0 t} . \tag{124}$$

This consideration is necessary only for $\lambda > 0$ in eq.(114); at $\lambda = 0$, we can put $\mathfrak{C} = \mathfrak{E}$. The collineatory transformation (123) is performed in j individual steps, in which case, for $\mu = 1, 2, \ldots, j$ the similarity transformation

$$\mathfrak{L}_\mu^{-1}\cdot{}^{\mu-1}\mathfrak{K}\cdot\mathfrak{L}_\mu = {}^{\mu}\mathfrak{K} \tag{125}$$

will produce a chain of matrices ${}^{\mu}\mathfrak{K}$ whose first link ($\mu = 0$) is formed by the matrix ${}^{0}\mathfrak{K} = \mathfrak{K}$ and whose last link is formed by the matrix ${}^{j}\mathfrak{K} = \mathfrak{K}^0$. In this case, each matrix ${}^{\mu}\mathfrak{K}$ will have the following form:



(126)

where the matrix ${}^{\mu}\vartheta^{\square} = \begin{bmatrix} \cdot & 1 \\ & \ddots \\ & & 1 \\ & & & 0 \end{bmatrix}$ has the order $(j - \mu)$, while the matrices ${}^{\mu}\vartheta_{\nu} = \begin{bmatrix} \cdot & 1 \\ & \ddots \\ & & 1 \\ & & & 0 \end{bmatrix}$ (for $\nu = 0, 1, \ldots, \lambda$) have the orders*

$$ {}^{\mu}m_{\nu} = \hat{m}_{\nu} + \mathrm{Min}(\mu, i_{\nu+1}) - \mathrm{Min}(\mu, i_{\nu}) . \tag{127} $$

For the quantities which occur in eq.(127) and are not yet defined, the following must be set

$$ \hat{m}_0 = i_0 = 0 , \quad \mathrm{Min}(\mu, i_{\lambda+1}) = \mu . \tag{128} $$

Obviously, ${}^{0}m_{\nu} = \hat{m}_{\nu}$. The matrices ${}^{\mu}\mathfrak{R}_{\nu}$, also occurring in eq.(126) and having ${}^{\mu}m_{\nu}$ columns and $j - \mu$ rows, contain a 1 for $\mu < i_{\nu}$ in the last row and in the $(1 + i_{\nu} - \mu)^{\mathrm{th}}$ column and also contain a 1 in the last row and first column for $i_{\nu} \leq \mu < i_{\nu+1}$, but otherwise only zeros. In the case $\mu \geq i_{\nu+1}$, the quantity ${}^{\mu}\mathfrak{R}_{\nu}$ is the zero matrix. Here, the undefined $i_{\lambda+1}$ constitutes no restriction.

Obviously, ${}^{j}\mathfrak{R} = \mathfrak{R}^{0}$ has the Jordan normal form while, for $\mu = 0$, the quantity ${}^{0}\mathfrak{R} = \mathfrak{R}$ according to eq.(114) has the form of eq.(126). It should be noted here that, because of ${}^{0}m_{0} = 0$, no matrix ${}^{0}\mathfrak{R}_{0}$ resp. ${}^{0}\vartheta_{0}$ occurs. In what follows, we will make an induction from $\mu - 1$ to μ. For this, we define a subscript $\ell(\mu)$ for each $\mu = 1, 2, \ldots, j$ in the following manner: If μ falls into one of the intervals with the end points $i_1, i_2, \ldots, i_{\lambda}$ (the left-hand end point is

* See the definitions at the beginning of Section 4.

included in the interval while the right-hand point is excluded), we will denote the left interval end point by $i_{\ell}(\mu)$. Thus,

$$i_{\ell(\mu)} \leq \mu < i_{\ell(\mu)+1} \quad \text{for} \quad i_1 \leq \mu < i_\lambda \,. \tag{129a}$$

For the not yet covered values of μ, we assume

$$\ell(\mu) = 0 \quad \text{for} \quad \mu < i_1 \tag{129b}$$

$$\ell(\mu) = \lambda \quad \text{for} \quad \mu \geq i_\lambda \,. \tag{129c}$$

The following relations, derived from eq.(127), should be noted: /103

$$j - \mu + \sum_{\nu=0}^{\lambda} {}^{\mu}m_\nu = j + \sum_{\nu=1}^{\lambda} \hat{m}_\nu = j + [\lambda] \,, \tag{130}$$

where eq.(21) had been taken into consideration, as well as

$$j - \mu + \sum_{\nu=0}^{\ell(\mu)} {}^{\mu}m_\nu = j + \sum_{\nu=0}^{\ell(\mu)} \hat{m}_\nu = j + [\ell(\mu)] \tag{131}$$

and

$${}^{\mu}m_\nu = \hat{m}_\nu \quad \text{for} \quad \nu > \ell(\mu) \,. \tag{132}$$

It is useful to repeat the matrix (126) in a more detailed form (for $\mu > 0$):

$${}^{\mu}\mathcal{A} = \begin{pmatrix} {}^{\mu}\mathcal{J}_0 & 0 & {}^{\mu}\mathcal{K}_{\ell(\mu)} & {}^{\mu}\mathcal{K}_{\ell(\mu)+1} & \cdots & {}^{\mu}\mathcal{K}_\lambda & 0 \\ {}^{\mu}\mathcal{J}_0 & & & & & & \\ & {}^{\mu}\mathcal{J}_1 & & & & & \\ & & \ddots & & & & \\ & & & {}^{\mu}\mathcal{J}_{\ell(\mu)-1} & & & \\ & & & & {}^{\mu}\mathcal{J}_{\ell(\mu)} & & \\ & & & & & {}^{\mu}\mathcal{J}_{\ell(\mu)+1} & \\ & & & & & & \ddots \\ & & & & & & {}^{\mu}\mathcal{J}_\lambda \\ & & & & & & \hat{\mathcal{K}}_{\lambda+1} \\ & & & & & & \ddots \\ & & & & & & \hat{\mathcal{K}}_\infty \end{pmatrix} \tag{133}$$

The matrices $\mathfrak{C}_\mu$ in eq.(125), to be constructed next, have a differing /104 structure depending on whether the indices $\mu - 1$ and μ belong to the same or to different intervals (129a, b, c).

First case: We have

$$\iota(\mu-1) = \iota(\mu) , \tag{134}$$

i.e., the following is valid:

$$i_{\iota(\mu-1)} = i_{\iota(\mu)} \leq \mu-1 < \mu < i_{\iota(\mu-1)+1} = i_{\iota(\mu)+1} \text{ for } i_1 \leq \mu-1 < \mu < i_2 \tag{135a}$$

or, in the two limiting cases,

$$0 \leq \mu-1 < \mu < i_1 \text{ for } \mu < i_1 , \tag{135b}$$

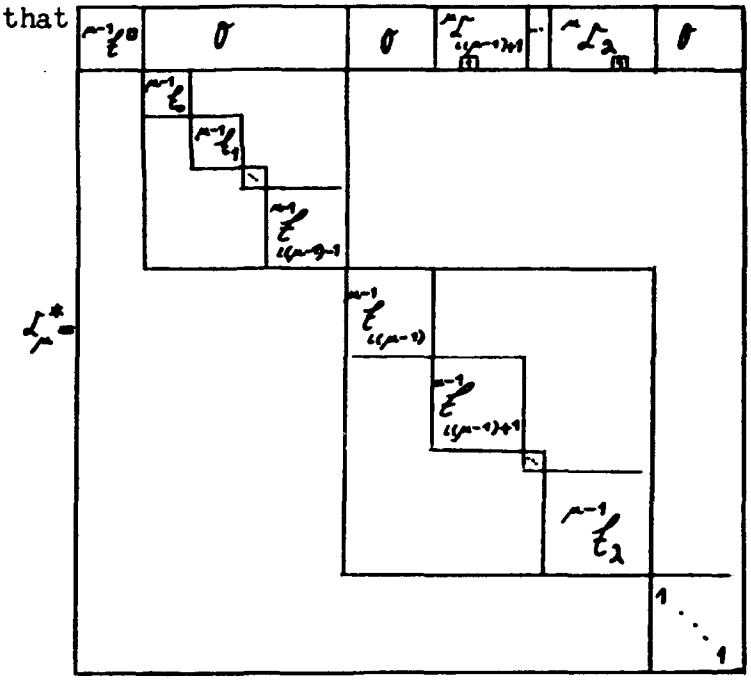
$$i_\lambda \leq \mu-1 < \mu \text{ for } \mu \geq i_\lambda . \tag{135c}$$

In this case, we put

$$\mathfrak{L}_\mu = \mathfrak{L}_\mu^* \cdot \mathfrak{W}_\mu , \tag{136}$$

where, at first, we assume that

$$\mathfrak{L}_\mu^* = \tag{137}$$



The orders of the unit matrices ${}^{\mu-1}\mathfrak{E}^{\square}$, ${}^{\mu-1}\mathfrak{E}_0$, ..., ${}^{\mu-1}\mathfrak{E}_\lambda$ correspond to the /105 orders of the matrices ${}^{\mu-1}\vartheta^{\square}$, ${}^{\mu-1}\vartheta_0$, ..., ${}^{\mu-1}\vartheta_\lambda$ in the matrix ${}^{\mu-1}\mathfrak{R}$ [see eqs.(126) or (133)]. The matrix ${}^{\mu}\mathfrak{C}_\nu$, as already shown in the matrix (137) is a null matrix if the matrix ${}^{\mu-1}\mathfrak{R}_\nu$ is also a null matrix. The remaining matrices ${}^{\mu}\mathfrak{C}_\nu$ for $\nu \geq \lambda$ are calculated on the basis of the transformation

$$ {}^{\mu-1}\mathfrak{L}_\nu = {}^{\mu-1}\mathfrak{E}_\nu \cdot {}^{\mu-1}\mathfrak{L}_\nu^{\top}, \tag{138} $$

from which it follows specifically that also the matrix ${}^{\mu}\mathfrak{C}_{\ell}(\mu - 1)$ is a zero matrix.

Relative to ${}^{\mu-1}\mathfrak{R}_\nu$ with $\nu > \ell(\mu - 1)$, the 1 in the last row in ${}^{\mu}\mathfrak{C}_\nu$ is shifted one place toward the left. In the case of ${}^{\mu-1}\mathfrak{RC}_\mu$, the ones in ${}^{\mu-1}\mathfrak{R}_\nu$ with $\nu > \ell(\mu - 1)$ are supplemented by a 1 which is shifted by one unit toward the left top. Since ${}^{\mu}\mathfrak{C}_\nu^{-1}$ is formed from ${}^{\mu}\mathfrak{C}_\nu$ by replacing the ones in the ${}^{\mu}\mathfrak{C}_\nu$ by -1, all ones in $\mathfrak{C}_\mu^{-1}\ {}^{\mu-1}\mathfrak{RC}_\mu$ in the $(j - \mu)^{\text{th}}$ row of ${}^{\mu-1}\mathfrak{RC}_\mu$ vanish except for the 1 in the first column of ${}^{\mu-1}\mathfrak{R}_\ell(\mu - 1)$ in the case of $\mu > 1$.

This means that the matrix $\mathfrak{C}_\mu^{*-1}\ {}^{\mu-1}\mathfrak{RC}_\mu^{*} = {}^{\mu}\mathfrak{R}^{*}$ has the property that the ones standing in the matrices ${}^{\mu-1}\mathfrak{R}_\nu$ for $\mu > \ell(\mu - 1)$ are shifted by one place toward the left top in a diagonal parallel to the main diagonal, while all other elementary components of the matrix ${}^{\mu-1}\mathfrak{R}$ are retained. Specifically, the one in the lower left corner of the matrix ${}^{\mu-1}\mathfrak{R}_\ell(\mu - 1)$ remains fixed for $\mu > 1$.

Next, we perform another similarity transformation with the "permutation matrix" $\mathfrak{B}_\mu$ which is produced from the n-row unit matrix, by supplementing /106 the rows and columns having the numbers

$$ j-\mu+1,\ j-\mu+2,\ \ldots,\ j-\mu+1+\sum_{\nu=0}^{\ell(\mu-1)-1} {}^{\mu-1}m_\nu = k_{\mu-1} $$

(the next following column exactly has the 1 in question) with the cyclic transformation matrix

$$ \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{139} $$

In the case $\mu < i_1$ [i.e., $\ell(\mu) = 0$], we must put $\mathfrak{B}_\mu = \mathfrak{E}$ because of the fact that no excess 1 need be eliminated here. It should be noted that

$$ \mathfrak{L}_\mu^{-1} = \mathfrak{L}_\mu^{\top} \tag{140} $$

In ${}^{\mu}\mathfrak{H}^{*} \cdot \mathfrak{B}_{\mu}$, the 1 which, until now, had been in the $(j - \mu + 1)^{th}$ row at the $(\eta_{\mu-1} + 1)^{th}$ place, has been shifted to the place directly diagonally above the still to be eliminated 1 which remains at its place; all matrices in the ${}^{\mu-1}\vartheta_{\nu}$, located within the range of the matrix (139), are shifted by one unit toward the left. In the case of $\mathfrak{B}_{\mu-1} \cdot {}^{\mu}\mathfrak{H}^{*} \cdot \mathfrak{B}_{\mu}$, the same ${}^{\mu-1}\vartheta_{\nu}$ are shifted by one unit toward the top while the 1 in the $(j - \mu)^{th}$ row remains unchanged; the 1 standing diagonally toward the lower left, which is to be eliminated, has been shifted in the same column into the last row of $\mathfrak{B}_{\mu}$, differing from the corresponding row of the unit matrix, and thus changes ${}^{\mu-1}\vartheta_{\ell(\mu-1)}$ by one first row and column to a ${}^{\mu}\vartheta_{\ell(\mu)}$ with an order greater by one [note: $\ell(\mu) = \ell(\mu - 1]$. Consequently, all in all the new similarity transformation with $\mathfrak{B}_{\mu}$ shifts the one, still standing at an unwanted place, by one place toward the left top, increases the order of ${}^{\mu-1}\vartheta_{\ell(\mu-1)}$ toward the left top by one, shifts the matrices ${}^{\mu-1}\vartheta_{0}, {}^{\mu-1}\vartheta_{1}, \ldots, {}^{\mu-1}\vartheta_{\ell(\mu-1)-1}$ by one place toward the left top, and finally decreases the order of ${}^{\mu-1}\vartheta^{\square}$ by one. This means that ${}^{\mu}\vartheta^{\square}$ receives the order $j - \mu$ while ${}^{\mu}\vartheta_{\ell(\mu)}$ receives the order ${}^{\mu}m_{\ell(\mu)} = {}^{\mu-1}m_{\ell(\mu-1)+1}$. The other orders are retained [see eq.(127)]. The ones in the $(j - \mu)^{th}$ row are now standing /107 as stipulated by eq.(133). Consequently, in this first case we have transformed the matrix ${}^{\mu-1}\mathfrak{H}$ with $\mathfrak{C}_{\mu} = \mathfrak{C}_{\mu}{}^{*}\mathfrak{B}_{\mu}$ in a similar manner into the matrix ${}^{\mu}\mathfrak{H}$.

Second case: In contrast to eq.(134), we assume

$$\ell(\mu-1) < \ell(\mu), \tag{141}$$

i.e.

$$i_{\ell(\mu-1)} \leqq \mu-1 < i_{\ell(\mu-1)+1} \leqq i_{\ell(\mu)} \leqq \mu < i_{\ell(\mu)+1} \text{ for } i_{1} \leqq \mu-1 < \mu < i_{2}, \tag{142a}$$

and, respectively,

$$0 \leqq \mu-1 < i_{1} \leqq \mu \quad \text{for} \quad \mu-1 < i_{1}, \tag{142b}$$

$$\mu-1 < i_{2} \leqq \mu \quad \text{for} \quad \mu \geqq i_{2}. \tag{142c}$$

Obviously, more accurately than eq.(141), we then have

$$\ell(\mu) = \ell(\mu-1) + 1 \quad \text{and} \quad i_{\ell(\mu)} - \mu = 0 . \tag{143}$$

As in the first case, we first use the similarity transformation with the matrix $\mathfrak{S}_\mu^* \cdot \mathfrak{B}_\mu$. As a result, we obtain the matrix

$$\overset{\mu}{*}\mathfrak{A} = \mathfrak{S}_\mu^{-1} \cdot \mathfrak{L}_\mu^{*-1} \cdot \overset{\mu-1}{}\mathfrak{A} \cdot \mathfrak{L}_\mu^{*} \cdot \mathfrak{S}_\mu , \tag{144}$$

in which the ones are standing in the $(j - \mu)^{th}$ row but [see also the second equation of the system (143)] are now, for $\nu = \ell(\mu) - 1$ as well as for $\nu = \ell(\mu)$, shifted to the first column of ${}_*^{\mu}\mathfrak{A}_\nu$. Consequently, the matrix ${}_*^{\mu}\mathfrak{A}$ differs from the matrix ${}^{\mu}\mathfrak{A}$ only by the one at the place $\ell(\mu) - 1$ in the $(j - \mu)^{th}$ row which is not present in ${}^{\mu}\mathfrak{A}$.

For eliminating this one, the following transformation is introduced:

$$\mathfrak{B}_{\ell(\mu)}^{-1} \cdot \overset{\mu}{*}\mathfrak{A} \cdot \mathfrak{B}_{\ell(\mu)} = \overset{\mu}{}\mathfrak{A} \tag{145}$$

with the "superposition matrix" $\mathfrak{B}_{\ell(\mu)}$ which is obtained from the n-rowed unit /108 matrix if a ${}^{\mu}m_{\ell(\mu)-1}$-rowed negative unit matrix is introduced there, as shown in the matrix (146):

$$\mathfrak{B}_{\ell(\mu)} = \begin{pmatrix} {}^{\mu}\ell_0 & & & & & & & \\ & {}^{\mu}\ell_0 & & & & & & \\ & & {}^{\mu}\ell_1 & & & & & \\ & & & \ddots & & & & \\ & & & & {}^{\mu}\ell_{\ell(\mu)-1} & & & \\ & & & & -{}^{\mu}\ell_{\ell(\mu)-1} & {}^{\mu}\ell_\mu & & \\ & & & & & & {}^{\mu}\ell_{\ell(\mu)+1} & \\ & & & & & & & \ddots \\ & & & & & & & {}^{\mu}\ell_2 \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \tag{146}$$

In this case, the unit matrices have the same orders as the corresponding matrices ${}^{\mu}\vartheta^{\square}$, ${}^{\mu}\vartheta_0$, ..., ${}^{\mu}\vartheta_\lambda$ in the matrix (133). The quantity $\mathfrak{B}^{-1}_{\ell(\mu)}$ is obtained by substituting in $\mathfrak{B}_{\ell(\mu)}$ the inserted matrix $-\mathfrak{E}_{\ell(\mu)-1}$ by $+\mathfrak{E}_{\ell(\mu)-1}$. Already ${}^{\mu}\mathfrak{R}$ has the prescribed new orders of the ϑ-matrices in ${}^{\mu}\mathfrak{R}$. Specifically, the order of ${}^{\mu-1}\vartheta_{\ell(\mu)}$ has increased by one while ${}^{\mu-1}\vartheta^{\square}$ has received an order lower by one. The application of $\mathfrak{B}_{\ell(\mu)}$ to ${}^{\mu}_{*}\mathfrak{R}$ eliminates the superfluous 1, but brings the /109 matrix $-{}^{\mu}\vartheta_{\ell(\mu)-1}$ to the same place at which $-\mathfrak{E}_{\ell(\mu)-1}$ is located in the matrix (146); application of $\mathfrak{B}^{-1}_{\ell(\mu)}$ to the left-hand side will again eliminate this auxiliary matrix so that we finally obtain ${}^{\mu}\mathfrak{R}^{*}$. Consequently, in this second case we must put

$$\mathcal{L}_\mu = \mathcal{L}_\mu^{*} \cdot \mathcal{U}_\mu \cdot \mathcal{B}_{\ell(\mu)}. \tag{147}$$

As the overall result of our considerations, we then obtain:

Theorem 12: The matrix $\mathfrak{R}$, by means of the similarity transformation (123) with

$$\mathcal{L} = \prod_{\mu} \mathcal{L}_\mu \tag{148}$$

is transformed into its Jordan normal form

$$\mathcal{K}^{\circ} = \begin{bmatrix} \mathcal{K}_0^{\circ} & & & \\ & \mathcal{K}_1^{\circ} & & \\ & & \ddots & \\ & & & \mathcal{K}_{\hat{\lambda}}^{\circ} \end{bmatrix} \tag{149}$$

Here, the matrices $\mathfrak{E}_\mu$ are defined by eqs.(136) resp. (147), with the auxiliary matrices (137), (146), and the described $\mathfrak{B}_\mu$. The elementary components $\mathfrak{N}^{\circ}_\nu$ for $0 \le \nu \le \lambda$ have the following orders m_ν [see eq.(127)]:

$$m_\nu = \hat{m}_\nu + \operatorname{Min}(j, i_{\nu+1}) - \operatorname{Min}(j, i_\nu), \tag{150}$$

where, accordingly, we must put $\nu = \lambda$ [see eq.(128)]

$$\operatorname{Min}(j, i_{\lambda+1}) = j, \tag{151}$$

for $\nu = 0$:

$$\hat{m}_0 = i_0 = 0 \tag{152}$$

---

* By means of such a superposition matrix $\mathfrak{B}$, the proof of theorem 7 can be conducted explicitly and simply, in a concise manner.

In the case of $m_0 = 0$, the elementary component $\mathfrak{R}_0^{\circ}$ will vanish. At $\nu > \lambda$, /110 i.e., if either $\lambda_\nu \neq 0$ or if no i_ν at all exists at $\alpha_\nu = 0$, the following is trivially valid:

$$m_\nu = \hat{m}_\nu \ , \quad \textit{specifically,} \quad m_0 = j \ . \tag{153}$$

Another remark should be made here as to the correlation of the rows s and $\hat{s}$. In accordance with eq.(17), s denotes the number of elementary components in the original Jordan normal form of $\mathfrak{R}$ so that s also will be the number of elementary components of eq.(149), which means that

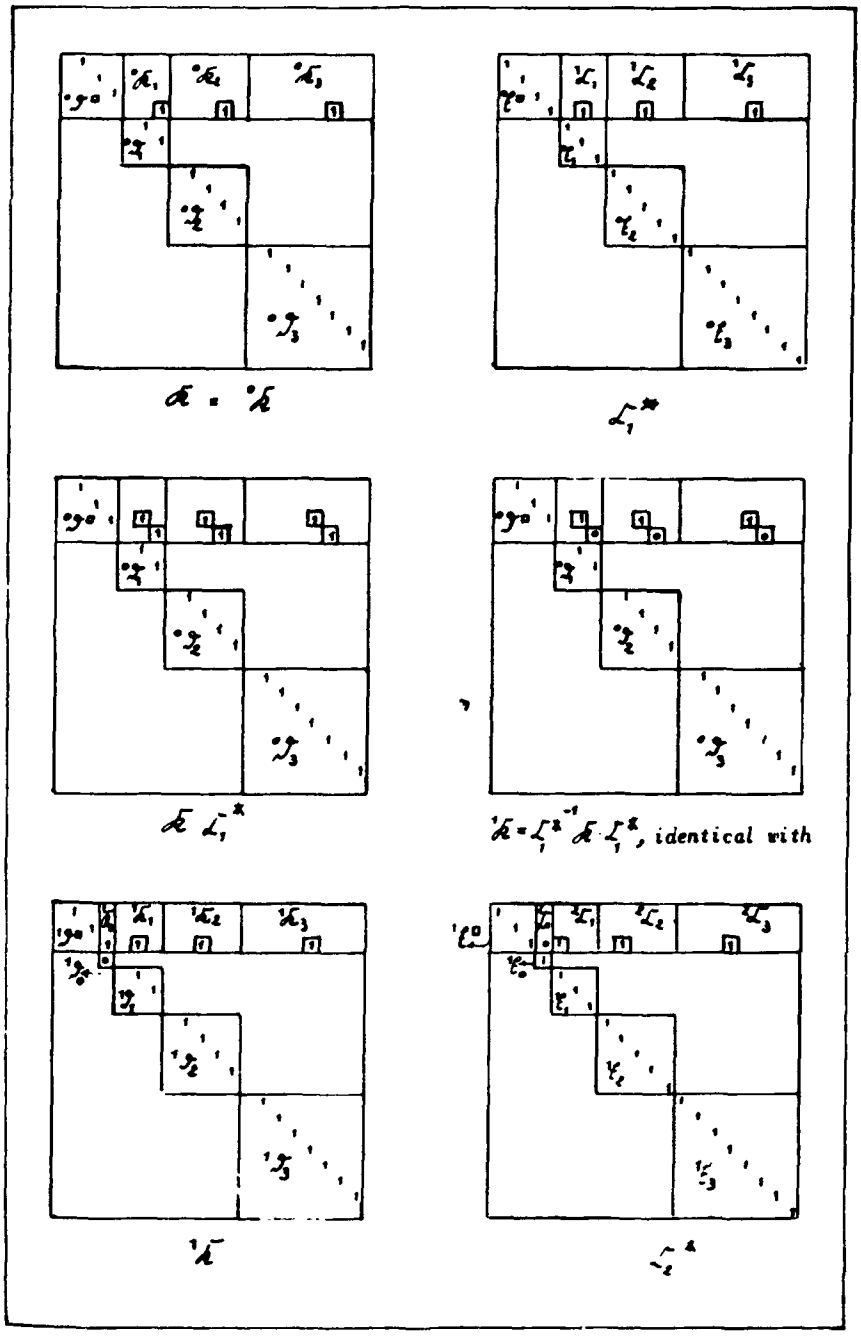
$$s = \hat{s} + \begin{cases} 1 \ , & \textit{if} \quad m_0 > 0 \\ 0, & \textit{if} \quad m_0 = 0 \ . \end{cases} \tag{154}$$

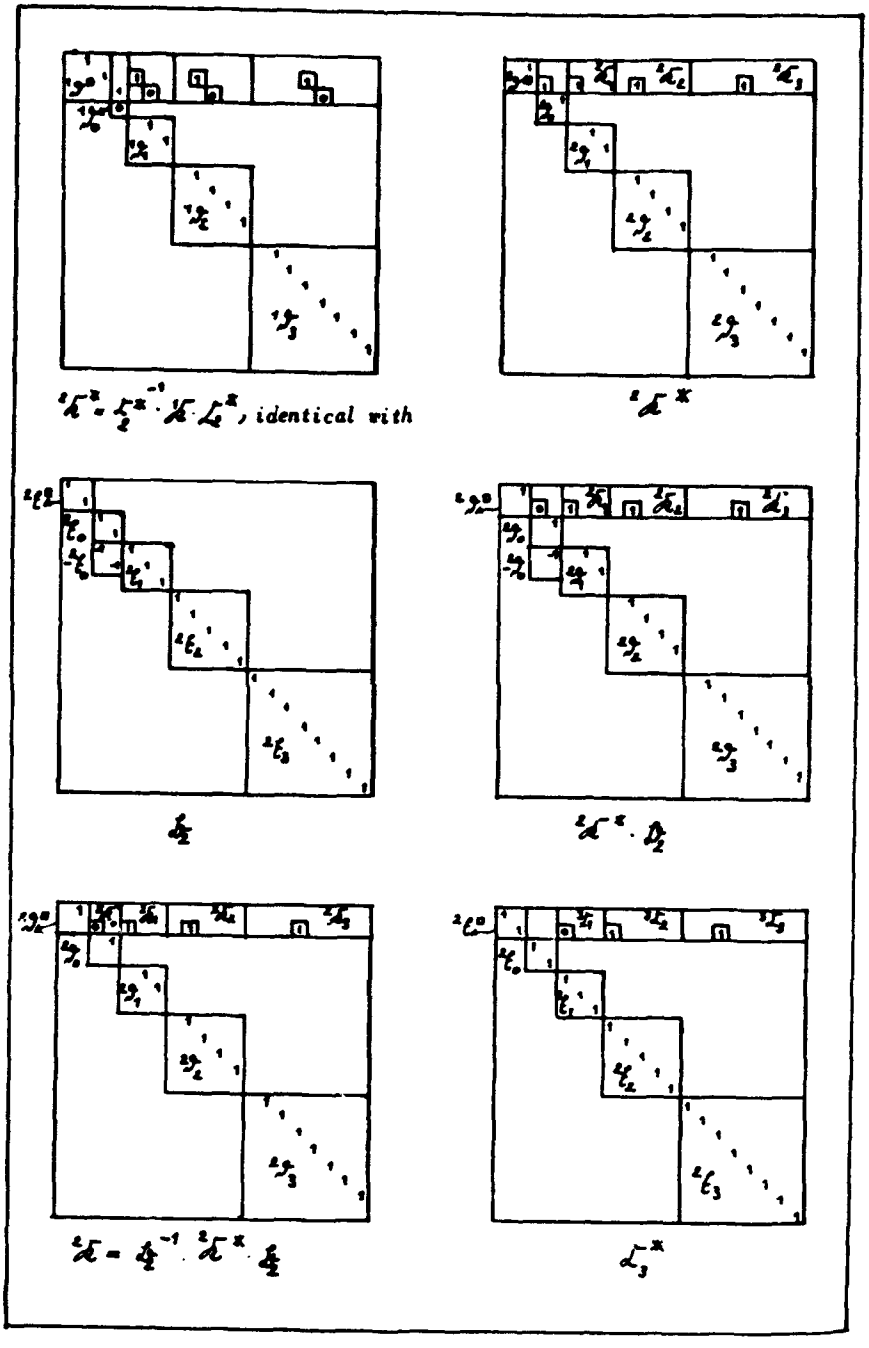
is valid. It is obvious (see theorem 5) that m_0 is zero if and only if the mean value (67) differs from zero for at least one $\hat{y}_{(\nu)}(t)$ ($\nu = 1, \ldots, \hat{\rho}$) since then we have, correspondingly, $i_\nu = 0$. In the special normal form (see the definition in Section 4), we will have $i_1 = 0$ according to eq.(74) and thus, according to eqs.(150) and (152), $m_0 = 0$. In the other cases ($i_1 > 0$, or absence of i_ν), we have $m_0 > 0$ [see eq.(153)].

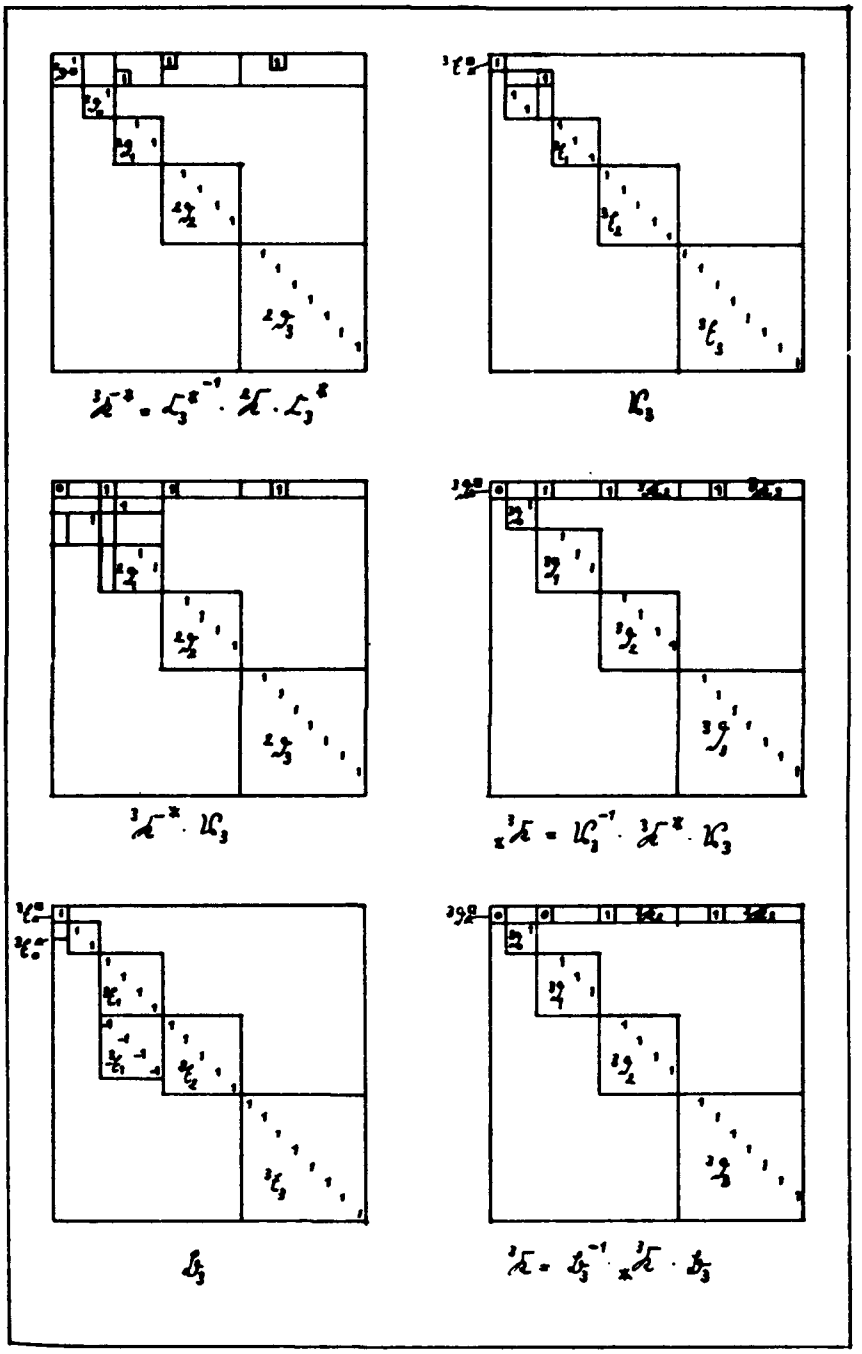
It is useful to give an explicit computation of the transformation of $\mathfrak{R}$ into the Jordan normal form, using a simple example in which all possibilities occur. Let

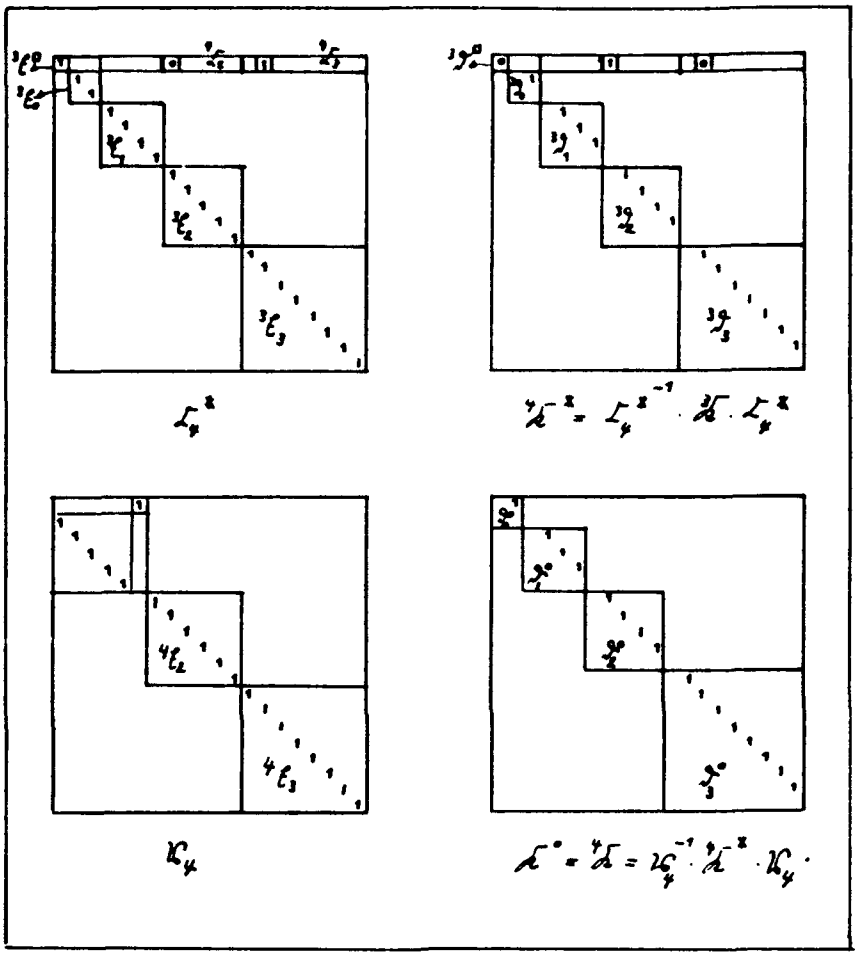
$$j=4, \ \hat{m}_1=3, \ \hat{m}_2=5, \ \hat{m}_3=8, \ n=20, \ i_1=2, \ i_2=3, \ i_3=5. \tag{155}$$

The general idea is demonstrated in the following matrices:









This completes the transformation of $\mathcal{A}$ into the Jordan normal form.

Section 7. Solutions of the Adjoint Homogeneous Differential Equation of the n^{th} Order

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The solution matrix $\mathfrak{M}^{\circ}(t)$ of eq.(6) conjugate to $\mathfrak{R}^{\vee}$ in eq.(149), can be calculated in accordance with eqs.(124) and (148). Since, however, the conditions for the resonance subcase and for the exceptional subcase [see eq.(28)] contain the solutions $\mathfrak{z}(t)$ of the adjoint homogeneous system (11) periodic with P [respectively of the adjoint homogeneous equation (9)], we will discuss this first. Analogous to eq.(124) and because of eq.(13), we have

$$\mathfrak{Z}^{\circ}(t) = \mathfrak{Z}(t) \cdot (\mathfrak{L}^{-1})^{T}. \tag{156}$$

We then again decompose the transformation $\mathfrak{T}$ into the j subtransformations $\mathfrak{T}_{\mu}$, according to eq.(148):

$$^{\mu}\mathfrak{Z}(t) = {}^{\mu-1}\mathfrak{Z}(t) \cdot (\mathfrak{L}_{\mu}^{-1})^{T} \tag{157}$$

after which, we will investigate the individual matrices $^{\mu}\mathfrak{z}(t)$ for $\mu = 0, 1, \ldots, j$, whose first and last matrices are

$$^{0}\mathfrak{Z}(t) = \mathfrak{Z}(t), \text{ resp. } {}^{j}\mathfrak{Z}(t) = \mathfrak{Z}^{\circ}(t) \tag{158}$$

We then partition the matrix $^{\mu}\mathfrak{z}(t)$, in accordance with the structure of $^{\mu}\mathfrak{R}$ in eq.(126), into $\hat{s} + 2$ submatrices

$$^{\mu}\mathfrak{Z}(t) = {}^{\mu}\mathfrak{Z}^{\square}(t) + \sum_{\nu=0}^{\hat{s}} {}^{\mu}\mathfrak{Z}_{\nu}(t), \tag{159}$$

where $^{\mu}\mathfrak{Z}^{\square}$, in addition to zero columns, contains only the first $j - \mu$ columns of $^{\mu}\mathfrak{Z}$, while $^{\mu}\mathfrak{z}_{0}$ contains the next $^{\mu}m_{0}$ columns, $^{\mu}\mathfrak{Z}_{1}$ the next following $^{\mu}m_{1}$ columns, and so on.

It is useful to introduce the following notation:

$$^{\mu}[[\nu]] = j - \mu + {}^{\mu}m_{0} + {}^{\mu}m_{1} + \cdots + {}^{\mu}m_{\nu}, \quad \mu = 0, 1, \ldots, j. \tag{160}$$

In the last step ($\mu = j$) we will omit the superscript [see eq.(150)]:

$$^{j}[[\nu]] = [[\nu]] = m_{0} + m_{1} + \cdots + m_{\nu} = \sum_{k=0}^{\nu} \hat{m}_{k} + \mathrm{Min}(j, i_{\nu+1}), \qquad \nu = 0, 1, \ldots, \hat{s}. \tag{161}$$

Then, the following theorem applies: /116

Theorem 13 The solutions $\hat{z}_{[\nu]}(t)$ ($\nu = 1, 2, \ldots, s$) of eq.(52) periodic with P are, in the same sequence, identical with the solutions $z_{[[\nu]]}(t)$ of eq.(9), again for $\nu = 1, 2, \ldots, \hat{s}$. Consequently,

$$z_{[\nu]}(t) = \hat{z}_{[\nu]}(t) \quad \text{for } \nu = 1,2,\ldots,\hat{s}. \tag{162}$$

Proof: In the matrix ${}^{0}\mathfrak{Z}(t)$ according to eq.(157), the solution vectors periodic with P and resulting from the periodic $\hat{z}_{[\nu]}(t)$ for $\nu = 1, \ldots, \hat{s}$, are exactly the $\mathfrak{z}_{[\nu]}(t)$ which means that they are standing exactly in the last column of the ${}^{0}\mathfrak{Z}_{\nu}(t)$ for $\nu = 1, 2, \ldots, \hat{s}$. A successive application, on the right-hand side, of the transformations described in the preceding Section $(\mathfrak{S}_{\mu}^{-1})^{T}$, $(\mathfrak{B}_{\mu}^{-1})^{T} = \mathfrak{B}_{\mu}$ and $(\mathfrak{B}_{\ell(\mu)}^{-1})^{T}$ (if they occur at all) to the matrix ${}^{\mu-1}\mathfrak{Z}(t)$, will transform the last column of each ${}^{\mu-1}\mathfrak{Z}_{\nu}(t)$ unchanged into the last column of ${}^{\mu}\mathfrak{Z}_{\nu}(t)$ ($\nu = 1, 2, \ldots, \hat{s}$). Consequently, also the last component of these columns will remain unaltered. By this inductive syllogism, the proof for eq.(162) is obtained.

In the case $m_0 > 0$, i.e., $i_1 > 0$ or nonexistent, a further solution of eq.(9) periodic with P will occur in accordance with eq.(154) and theorem 5. For this, the following is valid:

Theorem 14: In the case $m_0 > 0$, the auxiliary solution vector periodic with P is present in the last column of ${}^{j}\mathfrak{Z}_{0}(t)$; its last component which, consequently, must be denoted by $z_{[[0]]}(t)$, has the following form:

$$z_{[[0]]} = -\sum_{\nu=1}^{\hat{s}} \sum_{\mu=\ell(\nu)}^{[\nu]} \hat{\gamma}_{\mu}(t)\, {}_{j}\tilde{\varphi}_{\mu}(t) - \sum_{\nu=1}^{\lambda} \hat{\gamma}_{\ell(\nu)+i_{\nu}-1}\, {}_{j} \tag{163}$$

If no i_ν is present, i.e., for $\nu > \lambda$, the last sum naturally is omitted. /117

Proof: First, it should be remembered that the quantities ${}_{j}\tilde{\varphi}_{\mu}$, occurring in eq.(163), originate from the last (j^{th}) rows of the matrices $\tilde{\Phi}^{*}$ in eq.(105),

whereas the quantities $\hat{\psi}_\mu(t)$ originate in the last [$(n - j)^{th}$] row of the matrix [see eq.(60)]

$$\hat{\mathcal{L}} = (\hat{\Phi}^{-1})^T , \tag{164}$$

where the row index n - j has been omitted here. Since all these functions are periodic with P, this statement is valid also for the expression (163).

A column of ${}^1\mathfrak{Z}_0(t) = \mathfrak{Z}_0(t)$ occurs for the first time as the only column of ${}^1\mathfrak{Z}_0(t)$. Since this column, at increasing μ, always remains the last column of ${}^\mu\mathfrak{Z}_0(t)$ with $\mu \geq 1$, it follows - as in the proof of theorem 13 - that this column cannot change anymore. Consequently, we have only a single column vector ${}^1\mathfrak{z}_0(t)$ of ${}^1\mathfrak{Z}_0(t)$ which is located in the j^{th} column of ${}^1\mathfrak{Z}(t)$; its last component is the sought $z_{[[0]]}(t)$.

In the case $i_1 > 0$, formation of ${}^1\mathfrak{Z}(t) = \mathfrak{Z}(t) \cdot (\mathfrak{S}_1^{-1})^T$ will directly yield

$$\begin{aligned} z_{[[0]]}(t) &= {}_n z_j(t) - \sum_{\lambda=1}^{\lambda} z_{[[\mu-1]]+i_\mu}(t) = \\ &= {}_n z_j(t) - \sum_{\lambda=1}^{\lambda} \hat{z}_{(\mu)+i_\mu-1}(t) . \end{aligned} \tag{165}$$

It should be recalled that the last row index n or n - j had been omitted in the elements of $\mathfrak{Z}$ or $\hat{\mathfrak{Z}}$. In the case that no i_1 exists ($\mu > \lambda$), the second sum is eliminated.

According to eq.(13) and in analogy with eq.(64), we have

$$\mathfrak{Z}^T(t) = \mathfrak{Y}^{-1}(t) = \begin{bmatrix} e^{\lambda_0 t} & \mathfrak{y}^*(t) \\ 0 & \hat{\mathfrak{y}}(t) \end{bmatrix}^{-1} = \begin{bmatrix} e^{-\lambda_0 t} & \mathfrak{Z}^{*T}(t) \\ 0 & \hat{\mathfrak{Z}}^T(t) \end{bmatrix} \tag{166}$$

with /118

$$\hat{\mathfrak{Z}}^T = \hat{\mathfrak{y}}^{-1} \quad \text{and} \quad \mathfrak{Z}^{*T}(t) = e^{-\lambda_0 t} \mathfrak{y}^*(t) \hat{\mathfrak{Z}}^T(t) , \tag{167}$$

as is readily verified from the relation

$$\mathfrak{Z}^T \mathfrak{y} = \mathfrak{E} .$$

Taking the definition of $\mathfrak{y}^*(t)$ in eq.(64) into consideration and also considering that ${}_n z_j(t)$ is the element in the right-hand lower corner of $\mathfrak{Z}^{*\tau}(t)$, the following is obtained from eq.(167):

$$ {}_n z_j(t) = -\sum_{\nu=1}^{n-j} \hat{z}_\nu(t)\,\hat{Y}_\nu(t) = -\sum_{\nu=1}^{\hat{s}} \sum_{\mu=(\nu)}^{[\nu]} \hat{z}_\mu(t)\cdot\hat{Y}_\mu(t), \tag{168} $$

where [see eq.(64)]

$$ \hat{Y}_\mu(t) = \int \hat{y}_\mu(\tau)\,d\tau\ . \tag{169} $$

is valid. According to eq.(165) this yields the intermediate formula

$$ z_{[\nu]}(t) = -\sum_{\nu=1}^{\hat{s}} \sum_{\mu=(\nu)}^{[\nu]} \hat{z}_\mu(t)\,\hat{Y}_\mu(t) - \sum_{\nu=1}^{j} \hat{z}_{(\nu)+i_\nu-1}(t)\cdot \tag{170} $$

Denoting, by $c_{(\nu)+i\,\nu-1}$, a $\hat{m}_\nu$-component vector which, in addition to zeros, contains only an i in the $((\nu) + i_\nu - 1)^{th}$ component, the ν^{th} subsum of eq.(170) can be written as follows for $\nu \le \lambda$:

$$ T_\nu = -\sum_{\mu=(\nu)}^{[\nu]} \hat{z}_\mu(t)\,\hat{Y}_\mu(t) - \hat{z}_{(\nu)+i_\nu-1}(t) = $$

$$ = -\left(\hat{z}_{(\nu)},\dots,\hat{z}_{[\nu]}\right)\left\{\begin{bmatrix} \int \hat{y}_{(\nu)}(\tau)\,d\tau \\ \vdots \\ \int \hat{y}_{[\nu]}(\tau)\,d\tau \end{bmatrix} + \tau_{(\nu)+i_\nu-1}\right\}. $$

Taking into consideration the formula for the reduced differential equation, which is analogous to eq.(43) and, in its subdivision, analogous to eq.(65) as well as considering eqs.(103) and (104), we will obtain

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$$ T_\nu = -\left(\hat{Y}_{(\nu)},\dots,\hat{Y}_{[\nu]}\right)e^{-\hat{\alpha}_\nu^{\tau} t}\left\{e^{\hat{\alpha}_\nu^{\tau} t}\begin{bmatrix} \tilde{j}\tilde{\varphi}_{(\nu)} \\ \vdots \\ \tilde{j}\tilde{\varphi}_{[\nu]} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ t \\ \vdots \\ \frac{t^{\hat{m}_\nu - i_\nu}}{(\hat{m}_\nu - i_\nu)!} \end{bmatrix}\right\}, \tag{171} $$

where the 1 as the $((\nu) + i_\nu - 1)^{th}$ component of the last vector originates from $c_{(\nu)} + i_{\nu-1}$. This last vector, however, is exactly $e^{\hat{R}_\nu^T \cdot t} \cdot c_{(\nu)+i_{\nu-1}}$ so that we obtain further

$$T_\nu = -\left(\hat{\tau}_{(\nu)}, \ldots, \hat{\tau}_{[\nu]}\right)\left\{\begin{bmatrix} {}^j\tilde{\tau}_{(\nu)} \\ \vdots \\ {}^j\tilde{\tau}_{[\nu]} \end{bmatrix} + e_{(\nu)+i_\nu-1}\right\}$$
$$= -\sum_{\mu=(\nu)}^{[\nu]} \hat{\tau}_\mu \, {}^j\tilde{\tau}_{(\nu)} - \hat{\tau}_{(\nu)+i_\nu-1} \quad \text{for } \nu = 1, \ldots, \lambda. \tag{172}$$

In the case $\nu = \lambda + 1, \ldots, \hat{s}$, taking eq.(94) into consideration, the last vector in eq.(171) is omitted. This will yield

$$T_\nu = -\sum_{\mu=(\nu)}^{[\nu]} \hat{\tau}_\mu \, {}^j\tilde{\tau}_{(\nu)} \quad \text{for } \nu = \lambda+1, \ldots, \hat{s}. \tag{173}$$

From eqs.(172) and (173), we directly obtain eq.(163).

In passing, we note the following:

Theorem 15: The function $z_{[[0]]}(t)$, investigated in theorem 14, satisfies the inhomogeneous reduced adjoint differential equation

$$\tilde{\hat{L}}[\hat{u}] \equiv (-1)^{n-j} \hat{u}^{(n-j)} + (-1)^{n-j-1}(a_1\hat{u})^{(n-j-1)} + \cdots + a_{n-j}\hat{u} = 1. \tag{174}$$

Proof: In view of eqs.(52), (56), and (57), it merely must be demonstrated that a solution vector $\hat{u}$

$$\hat{u}' = -\hat{R}^T(t)\hat{u} - \eta(t) \tag{175}$$

with

$$\eta(t) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{176}$$

exists whose last $[(n - j)^{th}]$ component is $z_{[[0]]}(t)$. /120

The variational method for the constants [see (Bibl.1), footnote 9] furnishes the following relation as the general solution of eq.(175):

$$\hat{u}(t) = \hat{\zeta}(t) \cdot \left(\int \hat{\eta}^T(\tau)\eta(\tau)d\tau + c\right) \tag{177}$$

with an arbitrary constant vector c and, as the last component,

$$u_{n-j}(t) = \sum_{\nu=1}^{\hat{s}} \sum_{\mu=(\nu)}^{[\nu]} \hat{z}_{\mu}(t) \left(-\int \hat{g}_{\mu}(\tau) d\tau + c_{\mu} \right). \qquad (178)$$

According to the general theorems [see, for example (Bibl.1), eqs.(16) and (17) and the context there], eq.(175) has a solution vector periodic with P if the following applies for all solutions of eq.(55), periodic with P:

$$-\int_0^P \hat{g}_{(\nu)}(\tau) d\tau = 0 \quad \text{for } \nu = 1, 2, \ldots, \hat{s}.$$

This is exactly the condition for $m_0 > 0$ [see for example the remarks after eq.(154)]. Consequently, eq.(178) can then be determined as a function periodic with P. The lower limit in the integrals can be so defined that $c_{\mu} = 1$ applies for $\mu = (\nu) + i_{\nu} - 1$ ($\nu = 1, \ldots, \lambda$) while it has a value of zero everywhere else. Now, eq.(178) obviously coincides with eq.(170) which means that theorem 15 is proved.

By j differentiations of eq.(174), we again verify that $z_{[[0]]}(t)$, obtained from eq.(165) or eq.(170), is the solution of eq.(9).

Section 8. The Minimal Order of Magnitudes of the Solutions and their Derivatives for the Resonance Case /121

With respect to the resonance case for the differential equation (1), the following statements can be made: The adjoint homogeneous differential equation (9), as defined in the preceding Section, has the solutions $z_{[[0]]}$, $z_{[[1]]}$, ..., $z_{[[\hat{\rho}]]}$ periodic with P, where $z_{[[1]]}$, ..., $z_{[[\hat{\rho}]]}$ according to theorem 13 is identical with the solutions $\hat{z}_{[1]}$, ..., $\hat{z}_{[\hat{\rho}]}$ periodic with P of the adjoint homogeneous reduced differential equation (52) and $z_{[[0]]}$ occurs only in the case of $m_0 > 0$. Consequently, if the resonance subcase exists for the inhomogeneous reduced differential equation (50), for an index $\nu > 0$, i.e., if the following is valid [see eq.(28)]:

$$\int_0^P \hat{z}_{[\nu]}(\tau)\, f(\tau)\, d\tau = a_{[\nu]} \neq 0\,, \tag{179}$$

then the resonance subcase also will exist for the inhomogeneous differential equation (1) for the same index ν; consequently, we then have

$$\int_0^P z_{[[\nu]]}(\tau)\, f(\tau)\, d\tau = a_{[[\nu]]} = a_{[\nu]} \neq 0\,. \tag{180}$$

Speaking generally, we will denote any index ν, i.e., also $\nu = 0$, as a resonance index provided that the resonance subcase exists for this ν. Thus, we have the following: If ν is a resonance index of the reduced differential equation (50), then ν will also be the resonance index for the differential equation (1).

We must now differentiate between the following cases:

I. No resonance index exists.

II. $\nu = 0$ is no resonance index, but at least one resonance index $\nu > 0$ exists.

III. $\nu = 0$ is the only resonance index.

IV. $\nu = 0$ is a resonance index, and at least one index $\nu > 0$ exists which also is a resonance index.

The points I and II also contain cases in which the solution $z_{[[0]]}$ is not present, i.e., in which $i_1 = 0$ applies.

In the Case I, either the principal case or the exceptional case is involved for the reduced differential equation (50). For the differential equation (1), at $i_1 = 0$, the same case as for the reduced differential equation (50) is involved. At $i_1 > 0$, the exceptional case is present for eq.(1) since at least one solution $z_{[[0]]}$, periodic with P, of the adjoint homogeneous differential equation (9) exists. /122

In the Case II, the minimal order of the power increment $\hat{m}$ of $\hat{x}(t)$, according to theorem 2, is determined by

$$\hat{m} = \max_{(\nu \text{ Res.}\ \nu=1,\ldots,\hat{p})} (\hat{m}_\nu) , \tag{181}$$

while the analogous order of increment m of the solution x(t) of eq.(1) is determined by

$$m = \max_{(\nu \text{ Res.}\ \nu=1,\ldots,\hat{p})} (m_\nu) \tag{182}$$

with m_ν according to eqs.(150)ff, with the resonance indices being the same in both cases.

In the Case III, the resonance case is not present for the reduced differential equation (50) but is present for the differential equation (1). The resonance order for x(t) is defined as [see eqs.(150) and (152)]

$$m = m_0 = \text{Min}\ (j, i_1) , \tag{183}$$

where, if no i_1 occurs, eq.(151) must be taken into consideration.

In the Case IV, the resonance case exists for the reduced differential equation (50) as well as for the differential equation (1). The resonance order for x(t) is defined by

$$m = \max_{(\nu \text{ Res.}\ \nu=0,1,\ldots,\hat{p})} (m_\nu) . \tag{184}$$

In determining the maximum, the quantity m_0 can be disregarded if, for at least one resonance index from the interval $1 \leq \nu \leq \hat{p}$ one i_ν exists; in that case, the corresponding $m_\nu > m_0$ according to eq.(150) because of $\hat{m}_\nu \geq \hat{m}_1 > > rm(j, i_1)$; see also the definition at the beginning of Section 4. /123

Similar statements apply also to the derivatives $x'(t)$, $x''(t)$, ..., $x^{(j-1)}(t)$ if, in the cases I to IV, the index j in the k^{th} derivative $x^{(k)}(t)$ in eq.(150) for the m_ν is replaced by $(j - k)$. The given increment orders, in the case of resonance, are always the minimal orders. It is entirely possible that, for example, x(t) represents a solution with minimal order (with the index j), while the corresponding derivative $x'(t)$ has a higher order than the

minimal order valid for $j - 1$. Conversely, it could be that $x'(t)$ is a solution with the minimal order valid for $j - 1$, while the once integrated function $x(t) = \int x'(t)dt$ has an order which is higher than the minimal order valid for j. This is due to the fact that, for different indices $j - k$, no differing determination of the parameter constants is necessary if a solution with minimal order is to be obtained in each particular case.

For explaining our discussions, the following example is used:

Let $\hat{m}_1 = 3$, $\hat{m}_2 = 5$, $\hat{m}_3 = 8$, $\hat{m}_4 = 4$, $\hat{m}_5 = 10$

$i_1 = 2$, $i_2 = 3$, $i_3 = 5$; i_4 and i_5 do not exist.

Consequently, it follows that

$$\hat{s} = 5, \; s = 6, \; n-j = 30.$$

First, we again compile the formulas which are valid for the orders m_ν [see eqs.(150) - (153)]*:

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$$\begin{cases} m_0 = \text{Min}\,(j, i_1) \\ m_\nu = \hat{m}_\nu + \text{Min}(j, i_{\nu+1}) - \text{Min}(j, i_\nu) \quad (\text{for } \nu = 1, \ldots, \lambda - 1) \\ m_\lambda = \hat{m}_\lambda + j - \text{Min}\,(j, i_\lambda) \\ m_\nu = \hat{m}_\nu \quad (\text{for } \nu = \lambda + 1, \ldots, \hat{s}). \end{cases} \tag{185}$$

Accordingly, the following results are obtained for the new j-dependent orders:

$$m_0 = \text{Min}\,(2, j) = \begin{cases} j & \text{for } j < 2 \\ 2 & \text{for } j \geqq 2 \end{cases}$$

$$m_1 = 3 + \text{Min}\,(3, j) - \text{Min}\,(2, j) = \begin{cases} 3 & \text{for } j \leqq 2 \\ 4 & \text{for } j \geqq 3 \end{cases}$$

$$m_2 = 5 + \text{Min}\,(5, j) - \text{Min}\,(3, j) = \begin{cases} 5 & \text{for } j \leqq 3 \\ 6 & \text{for } j = 4 \\ 7 & \text{for } j \geqq 5 \end{cases}$$

$$m_3 = 8 + j - \text{Min}\,(5, j) = \begin{cases} 8 & \text{for } j \leqq 5 \\ j + 3 & \text{for } j \geqq 6 \end{cases}$$

$$m_4 = 4 + j - j = 4$$

$$m_5 = 10 + j - j = 10.$$

* If the definition (195) is used, eq.(185) can be made more rigorous by replacing the index λ by γ.

This results in the Table:

	j=o	1	2	3	4	5	6	7	8	$j \geq 6$
m_0	0	1	2	2	2	2	2	2	2	2
m_1	3	3	3	4	4	4	4	4	4	4
m_2	5	5	5	5	6	7	7	7	7	7
m_3	8	8	8	8	8	8	9	10	11	j+3
m_4	4	4	4	4	4	4	4	4	4	4
m_5	10	10	10	10	10	10	10	10	10	10

Of these numbers, according to eq.(184), the maximum must be formed at fixed j, for the indices for which the resonance subcase exists. /125

For example, if the resonance subcase exists for $\nu = 3$, then m will be at least equal to m_3. In any case, however, m_3 is the maximum for $j \geq 6$ so that $m = m_3 = j + 3$ applies for $j \geq 6$. This means that the minimal order m finally increases linearly with increasing j.

Conversely, if the exceptional subcase exists for $\nu = 3$, the quantity m will never be larger than $m_5 = 10$. If the exceptional subcase also exists for $\nu = 5$, the quantity m will never exceed $m_2 = 7$; and so on. It is easy to demonstrate that m retains a constant value if the exceptional case exists for $\nu = 3$, at least beginning with a certain index j.

In general, it is easy to confirm the following theorem on the basis of eq.(185):

Theorem 16: If the resonance subcase is present for the index ν with the greatest existing i_ν, i.e., for $\nu = \lambda$, the minimal order m, beginning with a certain index j, will increase linearly with j (see also the definition in Sect.4). Naturally, this index λ is already defined before the special normal form is established. Conversely, if the exceptional case is present for the

index $\nu = \lambda$, the minimal order m will remain constant in any case, beginning with a certain index j. In all other respects, the behavior of the minimal order m, as a function of j, follows from the formula (184): $m = m(j)$ is partially piecewise constant and partially increases linearly with the slope one.

For each index ν for which an i_ν exists, the pattern of a broken curve will be obtained according to eq.(185) for the minimal order m_ν as a function of j, which has a horizontal slope for $0 \leq j \leq i_\nu$ and for $i_{\nu+1} \leq j$, whereas it has the slope 1 for $i_\nu \leq j \leq i_{\nu+1}$.

$$\begin{cases} \text{For } 0 \leqslant j \leqslant i_\nu \text{ we have } m_\nu = \hat{m}_\nu , \\ \text{for } i_{\nu+1} \leqslant j \text{ we have } m_\nu = \hat{m}_\nu + (i_{\nu+1} - i_\nu) , \\ \text{for } i_\nu \leqslant j \leqslant i_{\nu+1} \text{ we have } m_\nu = \hat{m}_\nu + (j - i_\nu) , \end{cases}$$

where, if $i_{\nu+1}$ no longer exists ($\nu = \lambda$), the last law is valid for all $j \geq i_\nu$:

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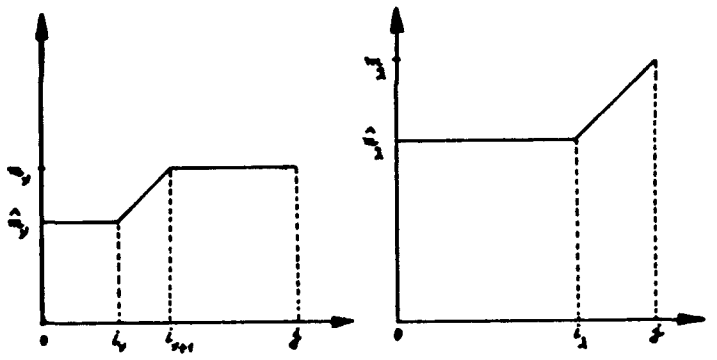

Fig.1

Since the following relation always is in question for two indices $\nu_2 > \nu_1$ with existing i_{ν_2} and i_{ν_1} [see eq.(74)]

$$\hat{m}_{\nu_2} - \hat{m}_{\nu_1} > i_{\nu_2} - i_{\nu_1}$$

the curve for the index ν_2 will begin with a constant which is greater than the end constant for the index ν_1, i.e., the curves for the various indices ν_1 and ν_2 with existing i_ν do not intersect. For each index ν for which no i_ν exists, $m_\nu = \hat{m}_\nu$ will be constant for $0 \leq j$.

Thus, the following statement is obtained for the minimal order according to eq.(184): Let $\bar{\nu}$ be the highest index ν with existing $i_{\bar{\nu}}$ for which the resonance case is present. In addition, let $\underline{\nu}$ be the highest resonance index at nonexisting $i_{\underline{\nu}}$. Then, the following obviously applies (see definition 3 in Sect.4):

$$m = \operatorname{Max}_{\substack{\nu \text{ res.} \\ \nu = 0,1,\ldots,\hat{\rho}}} (m_\nu) = \operatorname{Max} (m_{\bar{\nu}}, m_{\underline{\nu}}), \tag{186}$$

where the pattern of $m_{\bar{\nu}}$ as a function of j, yields a broken curve (see Fig.1), while $m_{\underline{\nu}}$ is a constant.

We would like to mention a few interesting relations: In the case that /127 only $\nu = 0$ is the resonance index of eq.(1), i.e., in the case III, at least one solution $\hat{x}(t)$ periodic with P of eq.(50) exists as we already know, whose mean value can be either equal to zero or different from zero. Then, the following theorem applies:

Theorem 17: If $i_1 = 0$, i.e., if no solution $z_{[[0]]}$ periodic with P of the adjoint homogeneous differential equation (9) exists, then - if at all - solutions $\hat{x}(t)$ of eq.(50) periodic with P will exist whose mean value differs from zero, as well as solutions whose mean value is equal to zero.

If $i_1 > 0$ or if no i_1 exists and if a solution $\hat{x}(t)$ periodic with P whose mean value differs from zero is present, then also every other solution $\hat{x}(t)$ periodic with P will have the same mean value differing from zero and $\nu = 0$ will be the resonance index. Conversely, if a solution $\hat{x}(t)$ of zero mean value and periodic with P exists, then also all other solutions $\hat{x}(t)$ periodic with P will have the mean value zero and $\nu = 0$ will be the exceptional index.

Proof: We can write the general solution $\hat{x}(t)$ of eq.(50), periodic with P, in the following form:

$$\hat{x}(t) = \hat{x}^*(t) + \sum_{1}^{\hat{\rho}} c_\nu \hat{y}_{(\nu)}(t) , \tag{187}$$

where $\hat{x}^*(t)$ is a special particular solution of eq.(50), periodic with P, while the sum next to it represents the general solution, periodic with P, for the homogeneous reduced differential equation (51).

In the case $i_1 = 0$, the quantity $\hat{y}_{(1)}$ will be a solution, periodic with P, of the homogeneous differential equation (50) with the mean value 1. The other solutions $\hat{y}_{(\nu)}$ ($\nu = 2, \ldots, \hat{\rho}$) of the homogeneous differential equation (50) have the mean value 0. This shows directly that, by a suitable selection of the constants c_1, the solution $\hat{x}(t)$ can be made to have a mean value of zero. Conversely, it is also possible to make certain that $\hat{x}(t)$ has a mean value /128 differing from zero.

For $i_1 > 0$ or for nonexistent i_1, all solutions $\hat{y}_{(\nu)}$, periodic with P, of the reduced homogeneous differential equation (51) will have the mean value zero. From this it follows that all solutions $\hat{x}(t)$, periodic with P, must have the same mean value as $\hat{x}^*(t)$ (the case $\hat{\rho} = 0$ is included here).

Then it merely remains to be demonstrated that

$$\text{from } \int_0^P \hat{x}(t)dt \text{ it follows } \left\{\begin{matrix} \neq 0 \\ \text{resp. } = 0 \end{matrix}\right\} \nu = 0 \quad \left\{\begin{matrix} \text{resonance} \\ \text{exceptional} \\ \text{index} \end{matrix}\right\} \tag{188}$$

We will demonstrate this indirectly: If, for the differential equation (1), the exceptional case were present, i.e., if $\nu = 0$ would be the exceptional index [the principal subcase cannot be present for $\nu = 0$ since the homogeneous differential equation (6) has at least the solution $y(t) \equiv 1$, periodic with P], a solution $x(t)$ of eq.(1) periodic with P would exist whose j^{th} derivative $\hat{x}^{(j)}(t) = x(t)$ is a not identically vanishing solution, periodic with P, of the reduced differential equation (50) with the mean value zero. However, this would mean that all solutions $\hat{x}(t)$, periodic with P, necessarily must have the mean value zero. If, consequently, $\int_0^P \hat{x}(t)dt \neq 0$, it follows necessarily that

$\nu = 0$ is the resonance index. If, conversely, $\int_0^P \hat{x}(t)dt = 0$, a solution $x(t)$ of the differential equation (1) can be obtained (see the auxiliary theorem in Sect.3) by j integrations of the function $\hat{x}(t)$, which is periodic with P and has a mean value zero. This means that the resonance case cannot exist for the differential equation (1) so that $\nu = 0$ must be the exceptional index since the main subcase had already been excluded above for $\nu = 0$. This proves theorem 17.

From this theorem, the following statements are obtained: Whenever the /129 quantity $\int_0^P z_{[[0]]}(t)\, f(t)dt$ differs from 0, i.e., whenever $\nu = 0$ is the resonance index, $\int_0^P \hat{x}(t)dt$ will differ from zero for a possibly existing periodic solution $\hat{x}(t)$ of eq.(50). Whenever $\int_0^P z_{[[0]]}(t)\, f(t)dt$ is equal to zero, i.e., whenever $\nu = 0$ is the exceptional index, the quantity $\int_0^P \hat{x}(t)dt$ will be equal to zero for a possibly existing periodic solution $\hat{x}(t)$ of eq.(50). Consequently, a relation must exist between the mean value of such an $\hat{x}(t)$ and the mean value of the function $z_{[[0]]}(t)f(t)$. Thus, the following theorem applies:

Theorem 18: If, in the case $i_1 > 0$ or nonexistent i_1, the reduced differential equation (50) possesses a solution $\hat{x}(t)$ periodic with P, the following equality will apply:

$$\int_0^P z_{[[0]]}(t)f(t)dt = \int_0^P \hat{x}(t)dt. \tag{189}$$

Proof: This can be proved by means of Lagrange's identity [see, for example (Bibl.4), Sect.5.3]

$$z_{[[0]]}\cdot\hat{L}[\hat{x}] - \hat{x}\cdot\bar{L}[z_{[[0]]}] = \frac{d}{dt}L^*[\hat{x}, z_{[[0]]}] =$$

$$= \sum_{\mu=0}^{n-j-1}\frac{d}{dt}\left[\hat{x}^{(n-j-\mu-1)}\left(a_\mu z_{[[0]]}\right) - \hat{x}^{(n-j-\mu-2)}\left(a_\mu z_{[[0]]}\right)' + \cdots + (-1)^{n-j-\mu-1}\hat{x}\cdot\left(a_\mu z_{[[0]]}\right)^{(n-j-\mu-1)}\right]$$

($a_0 = 1$) in the following manner:

Using [see eqs.(50) and (174)]

$$\hat{L}[\hat{x}] = f(t) \quad \text{and} \quad \overset{*}{L}[z_{[[0]]}] = 1$$

we obtain

$$z_{[[0]]}(t)\, f(t) - \hat{x}(t) = \frac{d}{dt}\, L^{*}\left[\hat{x}, z_{[[0]]}\right], \tag{190}$$

where $L^{*}[\hat{x},\ z_{[[0]]}]$, because of the periodicity $\hat{x}(t)$ and $z_{[[0]]}(t)$, is a function periodic with P whose derivative, obviously, has the mean value zero. From this, the argument is directly obtained.

Section 9. The Order of Magnitudes of the Derivatives of the Solutions in the Resonance Case

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In statements on the order of magnitude of the derivatives of the solutions $x(t)$ of eq.(1), for the resonance case, it is of importance whether the solutions $y_{((\nu))}(t) = \varphi_{((\nu))}(t)$ [see eq.(14)] of eq.(6), periodic with P, are constant for $\nu = 0, 1, \ldots, \hat{\rho}$ or whether they actually depend on t. Here, eq.(21) and thus also eq.(161) are analogously defined:

$$\begin{cases} ((\nu)) = m_0 + m_1 + m_2 + \ldots + m_{\nu-1} + 1 = \hat{m}_1 + \hat{m}_2 + \ldots + \hat{m}_{\nu-1} + \\ \quad + \operatorname{Min}(j, i_\nu) + 1 = (\nu) + \operatorname{Min}(j, i_\nu). \end{cases} \tag{191}$$

For this reason, let us start with an auxiliary consideration which describes the transition of the matrix $\Phi(t)$ from eq.(108) to the matrix $\Phi^{0}(t)$ from eq.(124). Primarily, according to eqs.(124), (14), (92), (113), and (123), we have

$$\begin{aligned} \Psi^{\circ}(t) &= \Phi^{\circ}(t)\, e^{\mathfrak{K}^{\circ} t} = \Psi(t)\, \mathfrak{L} = \Phi(t)\, e^{\mathfrak{K} t}\, \mathfrak{L} = \\ &= \Phi(t)\, \mathfrak{L}\, \mathfrak{L}^{-1}\, e^{\mathfrak{K} t}\, \mathfrak{L} = \Phi(t)\, \mathfrak{L}\, e^{\mathfrak{L}^{-1} \mathfrak{K} \mathfrak{L} t} = \\ &= \Phi(t)\, \mathfrak{L}\, e^{\mathfrak{K}^{\circ} t}, \end{aligned}$$

i.e.,

$$\Phi^{\circ}(t) = \Phi(t)\, \mathfrak{L} \tag{192}$$

with $\mathfrak{L}$ from eq.(148). In the elements of the column blocks ${}^{\nu}\Phi(t)$ with $\nu = 1$,

..., λ of $\Phi(t)$ from eq.(108), the partition is made analogously to eq.(101). Then, we can write

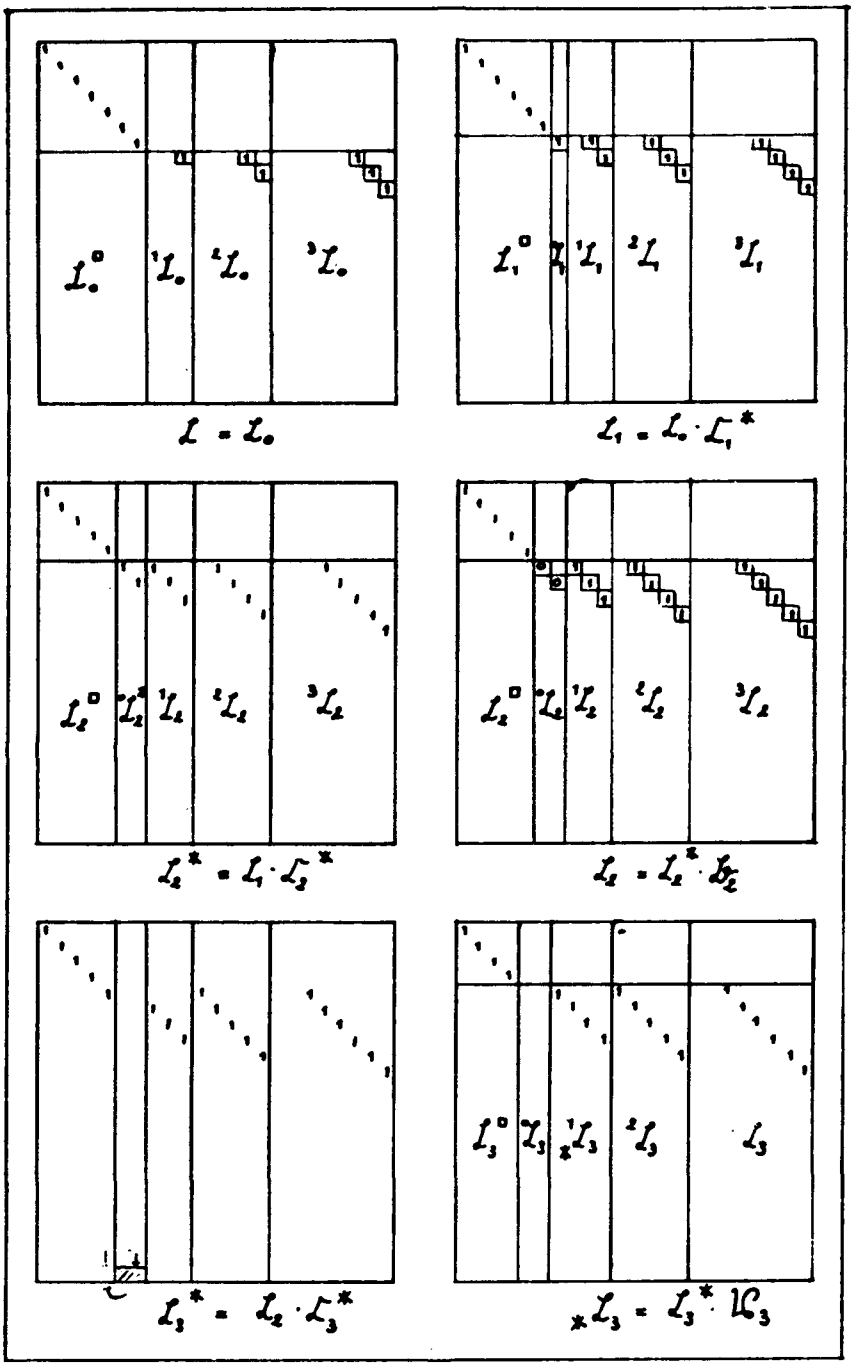
$$\Phi^{(\mu)} = \tilde{\Phi}^{(\mu)} + \mathcal{L} , \tag{193}$$

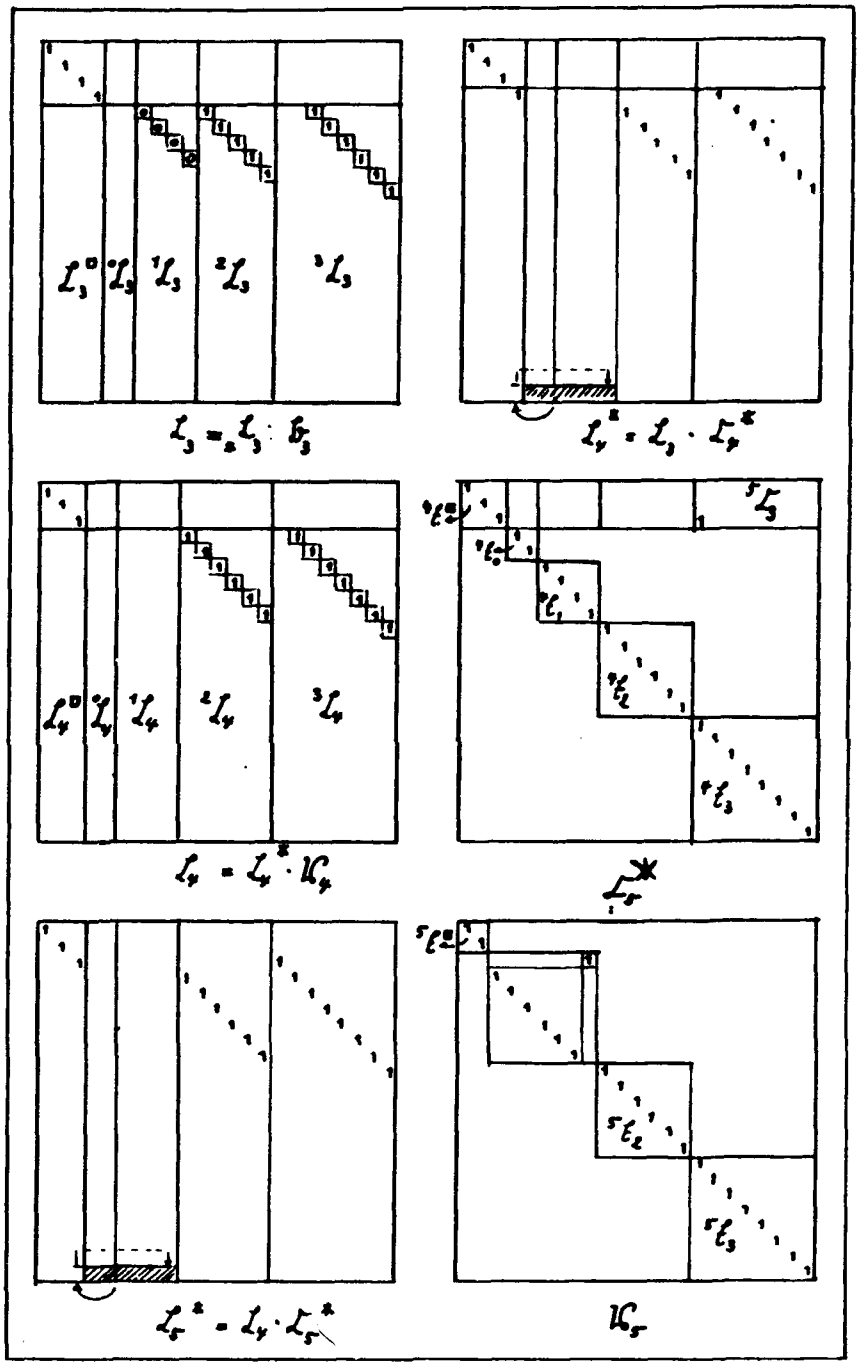
where the matrix $\tilde{\Phi}(t)$ contains only zeros in the columns 1, ..., j, and contains ${}_k\tilde{\varphi}_i(t)$ of zero mean value in the columns j + 1, ... n; the matrix $\mathfrak{L}$ contains only ones, etc., in the first j places of the main diagonal and, on partitioning (in a readily understandable symbolism) /131

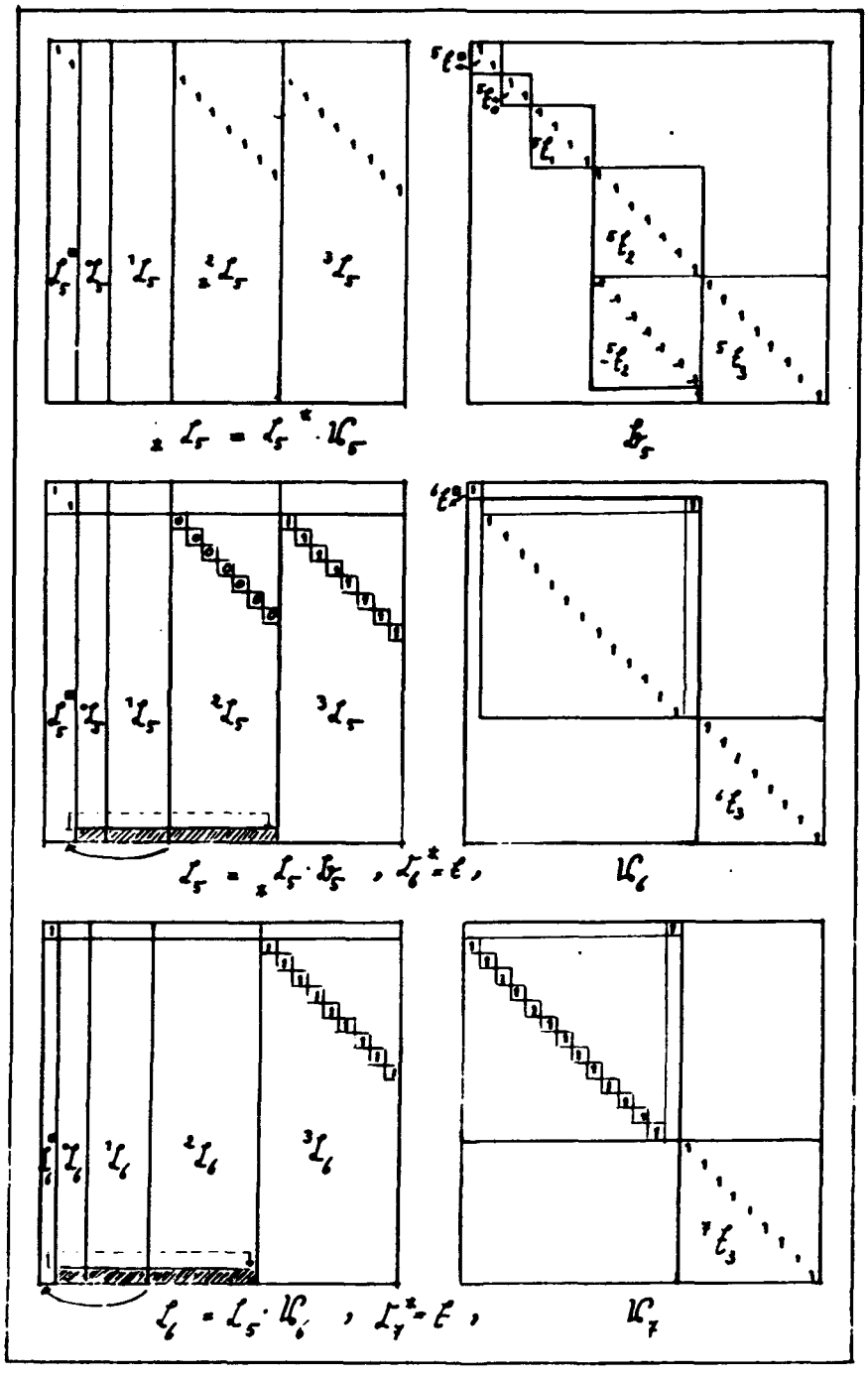
$$\mathcal{L} = \mathcal{L}^{0} + \sum_{\nu=1}^{\lambda} {}^{\nu}\mathcal{L} , \tag{194}$$

into ${}^{\nu}\mathfrak{L}$, it will also contain ones in the diagonal below -45° beginning at the element ${}^{\nu}l_{j+1,(\nu)+i\nu}$. For this concept, it is necessary to consider not only eq.(105) but also the fact that the rows of $\mathfrak{N}(t)$ result from each other by successive differentiation [see eq.(8)]. An application of the transformation matrix $\mathfrak{C}$ to $\tilde{\Phi}(t)$ furnishes, in all columns, always only elements of zero mean value. For this reason, the effect of the individual partial transformations (148) and (147) on $\mathfrak{L}$ will be investigated first.

For greater clarity, we will do this on the example (155) given at the end of Sect.6, with the exception that now j = 7 (instead of j = 4) is used so that all imaginable cases [see eq.(185), specifically the third relation] will be covered by this example. The matrices $\mathfrak{C}_{\mu}^{*}$, $\mathfrak{B}_{\mathfrak{l}(\mu)}$ and $\mathfrak{B}_{\mu}$, constructed earlier, need now be only supplemented by 7 - 4 = 3 ones, placed in front of the main diagonal. The following 19 matrices perform the stepwise transformation of $\mathfrak{L}$ into $\mathfrak{L}^{0} = \mathfrak{L}_{7}$.







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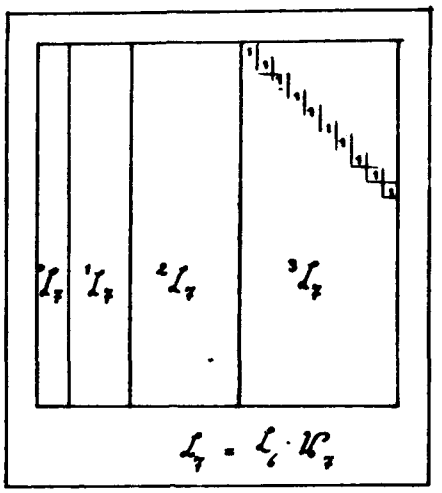

It is also useful to perform a stepwise transformation of the matrix $\tilde{\Phi}(t)$ into $\tilde{\Phi}^{o}(t)$ on hand of the same examples. This will be done in the following /136 11 matrices where (except for a few obvious exceptions) only the column indices of the functions ${}_{1}\tilde{\Phi}_{\nu}(t) = \tilde{\Phi}_{\nu}(t)$ occurring in eq.(101) are given, which at first appear in the $(j + 1)^{th}$ row for $\tilde{\Phi}(t) = \tilde{\Phi}_{o}(t)$.

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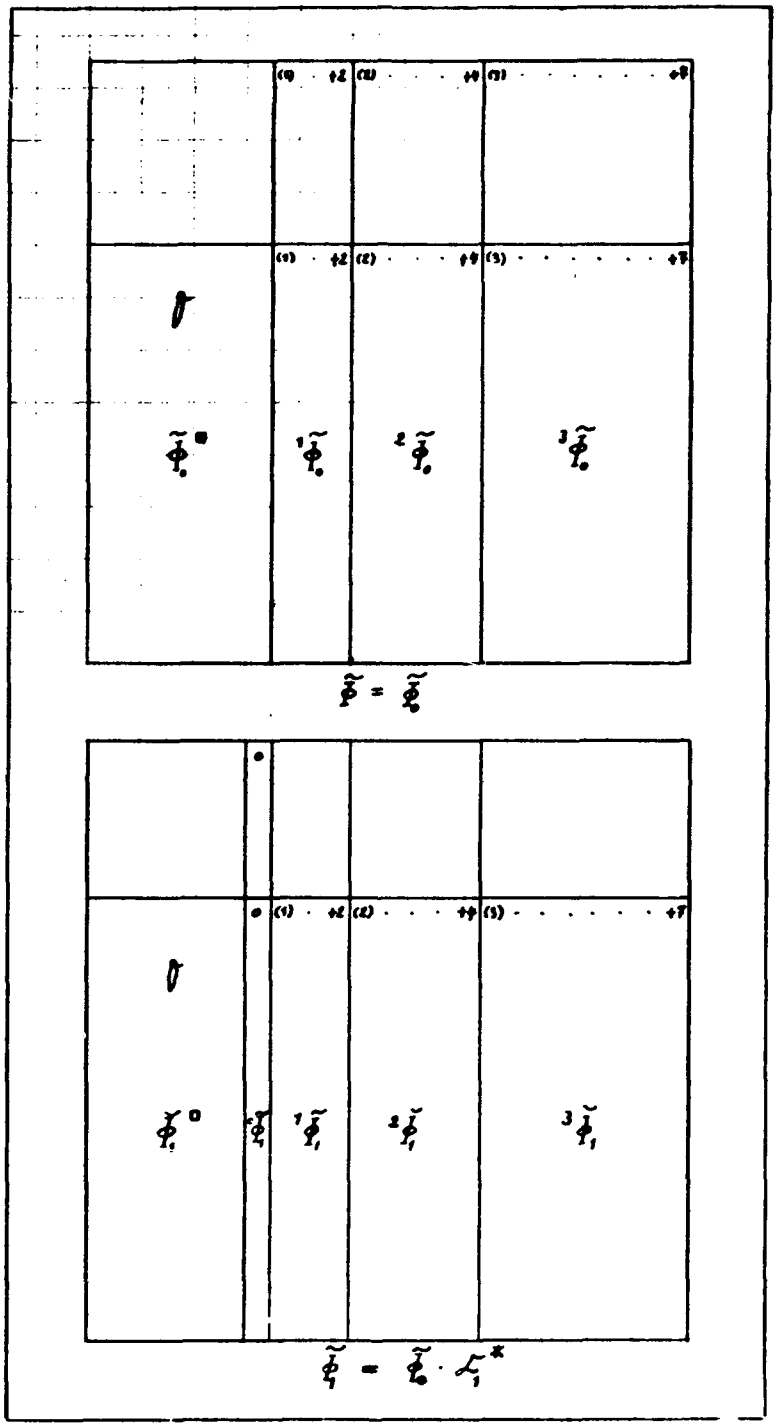

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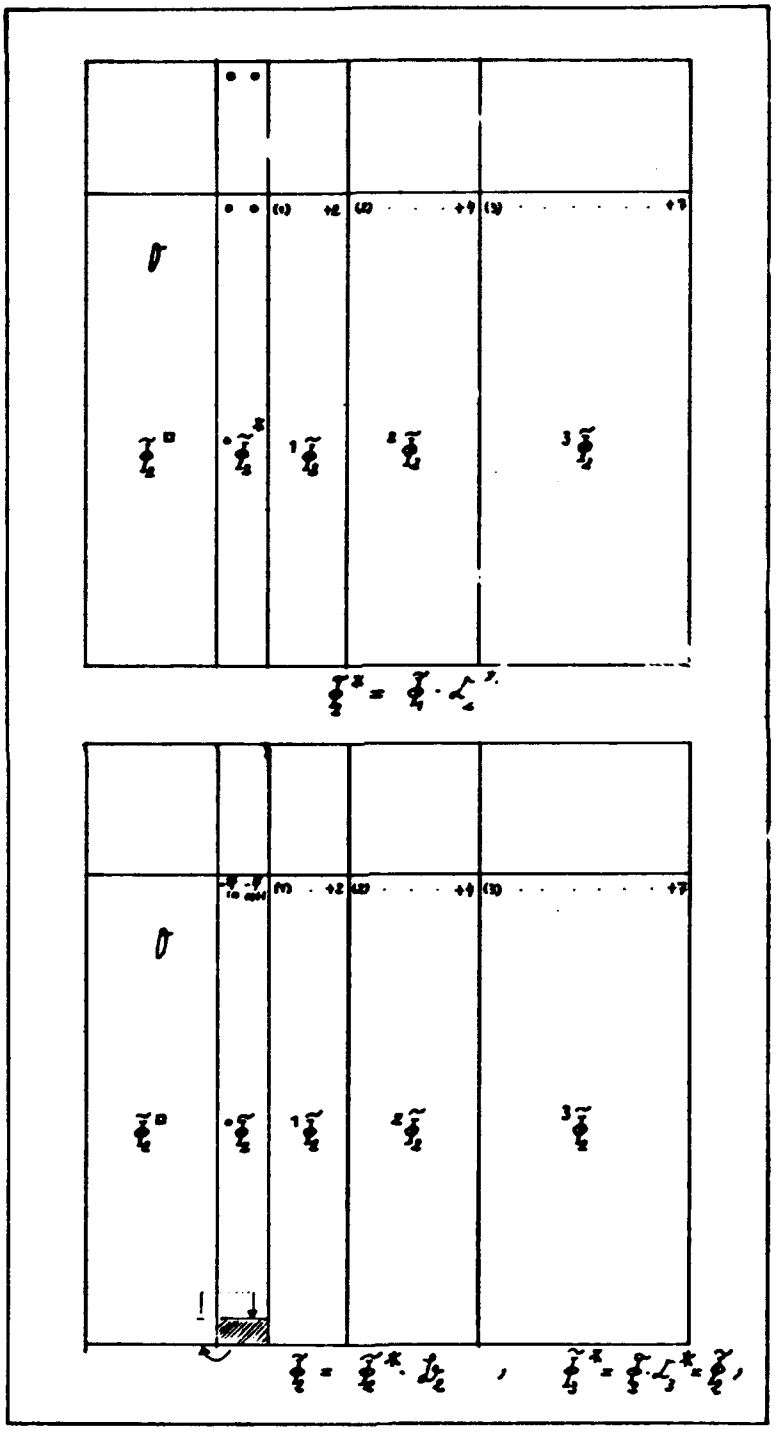

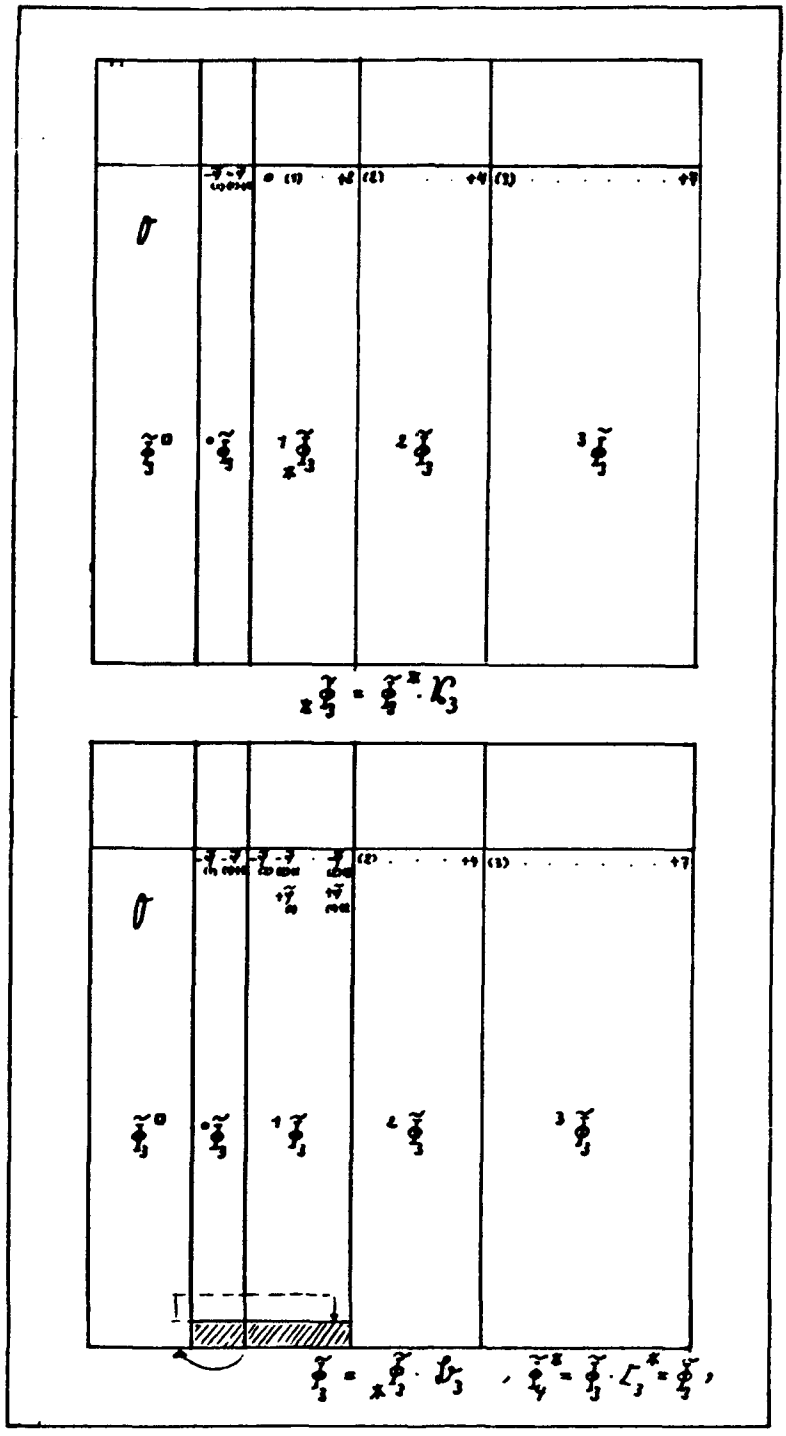

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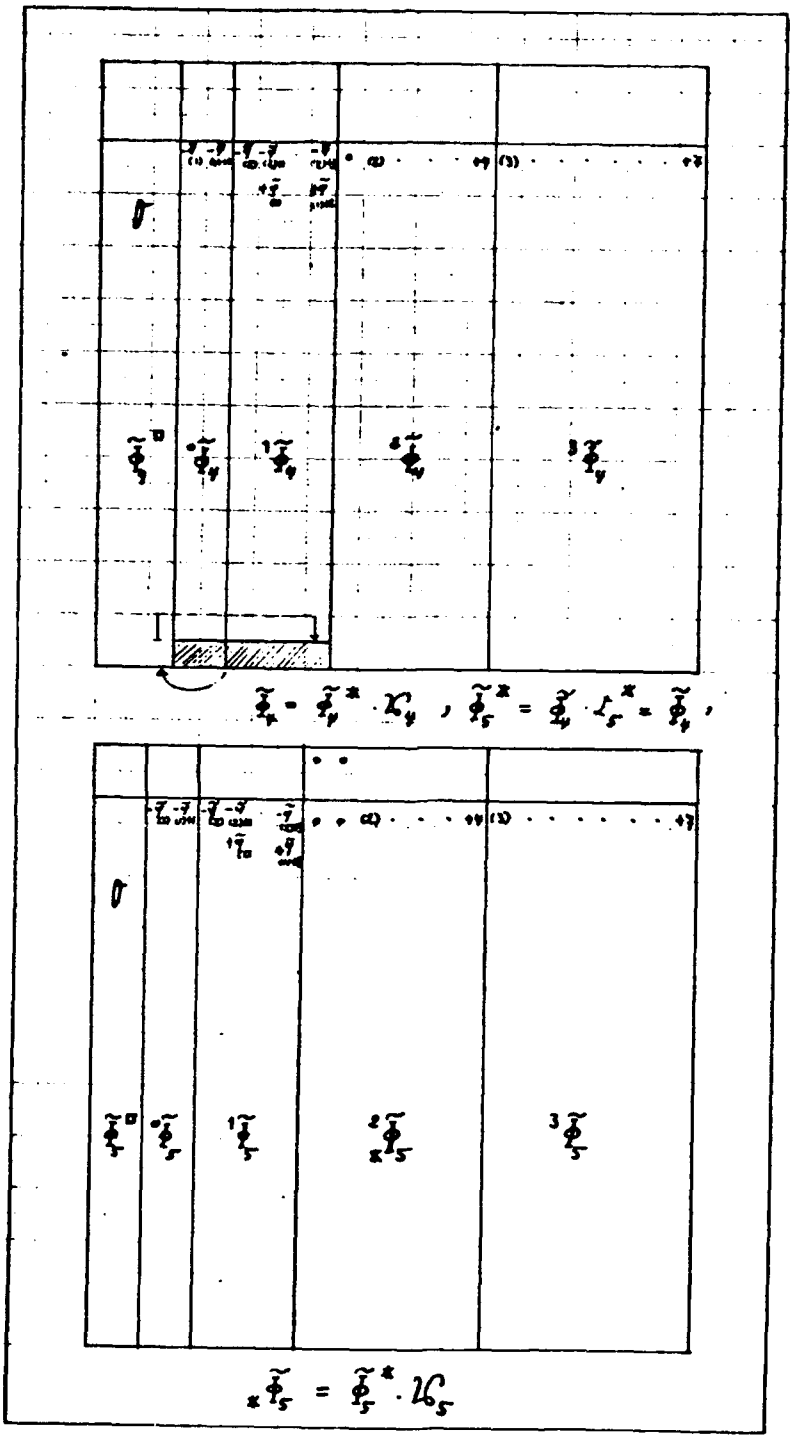

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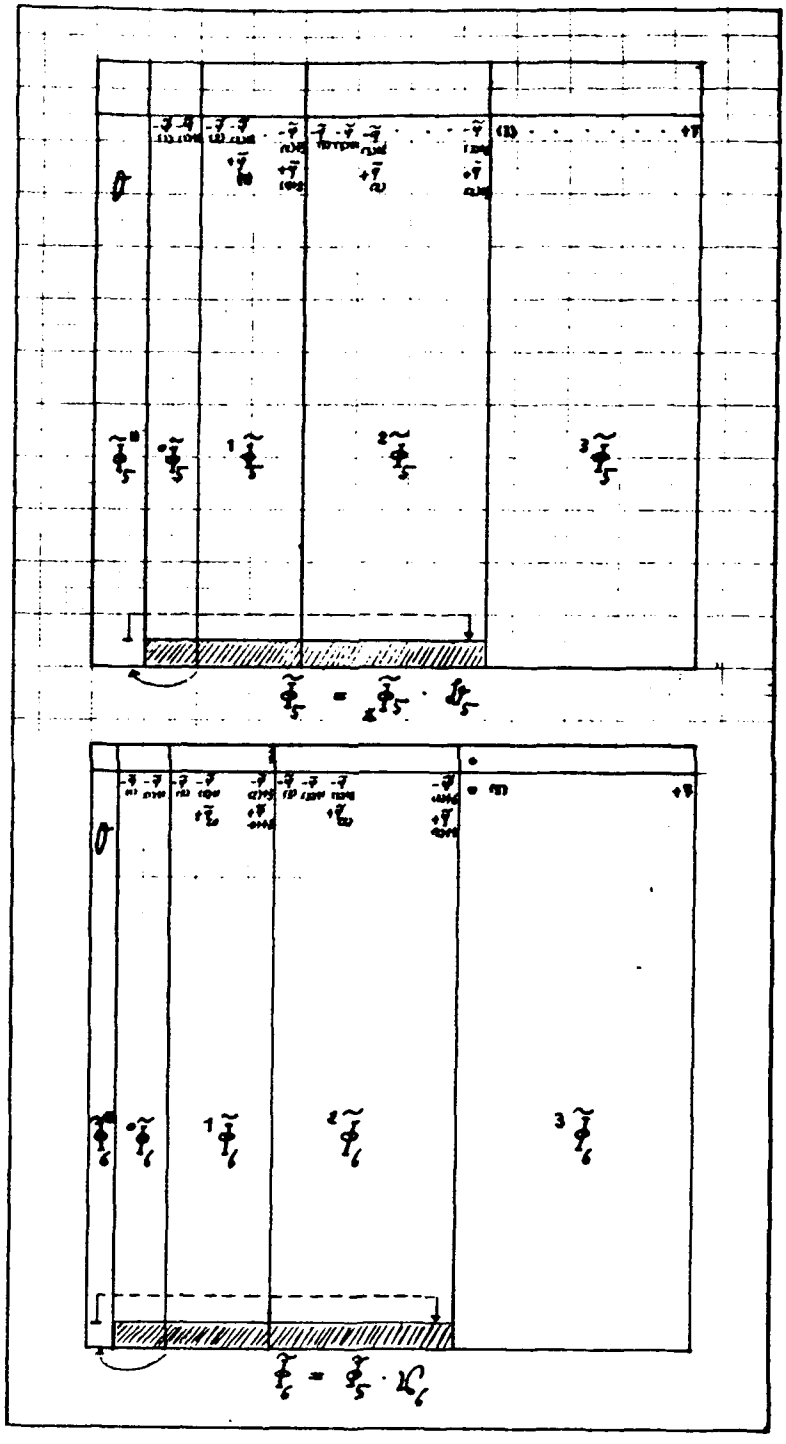

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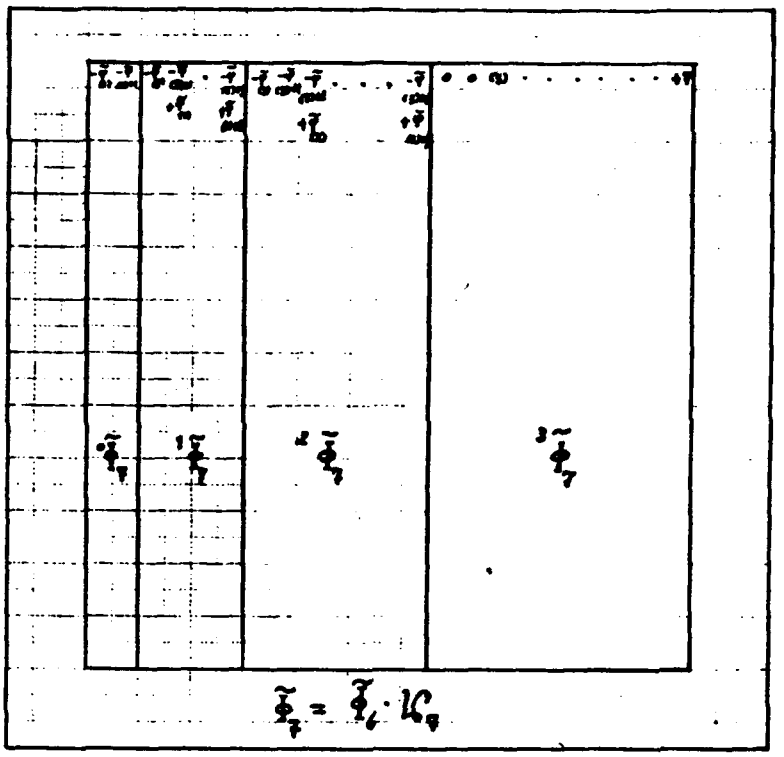

Let the first row of the matrix $\Phi_j = \tilde{\Phi}_j + \Omega_j$ again be written explicitly for the cases j = 1, 2, 4, and 7, i.e., $\gamma = 0, 1, 2$, and 3; $\lambda = 3$; $i_1 = 2$; $i_2 = 3$; $i_3 = 5$: /143

$$j=1\left\|\left(1\left|\tilde{\varphi}_{(1)}, 1+\tilde{\varphi}_{(1)+1}, \tilde{\varphi}_{(1)+2}\right|\tilde{\varphi}_{(2)}, \tilde{\varphi}_{(2)+1}, 1+\tilde{\varphi}_{(2)+2}, \tilde{\varphi}_{(2)+3}, \tilde{\varphi}_{(2)+4}\right|\right.$$
$$\left.\tilde{\varphi}_{(3)}, \tilde{\varphi}_{(3)+1}, \tilde{\varphi}_{(3)+2}, \tilde{\varphi}_{(3)+3}, 1+\tilde{\varphi}_{(3)+4}, \tilde{\varphi}_{(3)+5}, \tilde{\varphi}_{(3)+6}, \tilde{\varphi}_{(3)+7}\right)$$

$$j=2\left\|\left(-\tilde{\varphi}_{(1)}, -\tilde{\varphi}_{(1)+1}\right|1+\tilde{\varphi}_{(1)}, \tilde{\varphi}_{(1)+1}, \tilde{\varphi}_{(1)+2}\left|\tilde{\varphi}_{(2)}, 1+\tilde{\varphi}_{(2)+1}, \tilde{\varphi}_{(2)+2}, \tilde{\varphi}_{(2)+3}, \tilde{\varphi}_{(2)+4}\right|\right.$$
$$\left.\tilde{\varphi}_{(3)}, \tilde{\varphi}_{(3)+1}, \tilde{\varphi}_{(3)+2}, 1+\tilde{\varphi}_{(3)+3}, \tilde{\varphi}_{(3)+4}, \tilde{\varphi}_{(3)+5}, \tilde{\varphi}_{(3)+6}, \tilde{\varphi}_{(3)+7}\right)$$

$$j=4\left\|\left(-\tilde{\varphi}_{(1)}, -\tilde{\varphi}_{(1)+1}\right|-\tilde{\varphi}_{(2)}, -\tilde{\varphi}_{(2)+1}+\tilde{\varphi}_{(1)}, -\tilde{\varphi}_{(2)+2}+\tilde{\varphi}_{(1)+1}, -\tilde{\varphi}_{(2)+3}+\tilde{\varphi}_{(1)+2}\left|1, \tilde{\varphi}_{(2)}\right.\right.,$$
$$\left.\left.\tilde{\varphi}_{(2)+1}, \tilde{\varphi}_{(2)+2}, \tilde{\varphi}_{(2)+3}, \tilde{\varphi}_{(2)+4}\right|\tilde{\varphi}_{(3)}, 1+\tilde{\varphi}_{(3)+1}, \tilde{\varphi}_{(3)+2}, \tilde{\varphi}_{(3)+3}, \tilde{\varphi}_{(3)+4}, \tilde{\varphi}_{(3)+5}, \tilde{\varphi}_{(3)+6}, \tilde{\varphi}_{(3)+7}\right)$$

$$j=7\left\|\left(-\tilde{\varphi}_{(1)}, -\tilde{\varphi}_{(1)+1}\right|-\tilde{\varphi}_{(2)}, -\tilde{\varphi}_{(2)+1}+\tilde{\varphi}_{(1)}, -\tilde{\varphi}_{(2)+2}+\tilde{\varphi}_{(1)+1}, -\tilde{\varphi}_{(2)+3}+\tilde{\varphi}_{(1)+2}\left|-\tilde{\varphi}_{(2)}\right.\right.,$$
$$-\tilde{\varphi}_{(2)+1}, -\tilde{\varphi}_{(2)+2}+\tilde{\varphi}_{(2)}, -\tilde{\varphi}_{(2)+3}+\tilde{\varphi}_{(2)+1}, -\tilde{\varphi}_{(2)+4}+\tilde{\varphi}_{(2)+2}, -\tilde{\varphi}_{(2)+5}+\tilde{\varphi}_{(2)+3},$$
$$\left.-\tilde{\varphi}_{(2)+6}+\tilde{\varphi}_{(2)+4}\left|1, \vartheta_2, \tilde{\varphi}_{(3)}, \tilde{\varphi}_{(3)+1}, \tilde{\varphi}_{(3)+2}, \tilde{\varphi}_{(3)+3}, \tilde{\varphi}_{(3)+4},\right.\right.$$
$$\left.\left.\tilde{\varphi}_{(3)+5}, \tilde{\varphi}_{(3)+6}, \tilde{\varphi}_{(3)+7}\right)\right.$$

Since, in the general case, the considerations are completely analogous to this example, we can formulate the following theorem:

Theorem 19: If j is located in the interval

$$i_\gamma \leqq j < i_{\gamma+1} \quad (\gamma = 0, 1, \ldots, \lambda;\ i_0 = 0,\ i_{\lambda+1} \text{ is no limit}), \tag{195}$$

the following is valid [see eq.(97)]:

$$\varphi_{((\nu))}^{(t)} = \begin{cases} -\tilde{\varphi}_{(\nu+1)}^{(t)} & \text{for } \nu = 0, 1, \ldots, \gamma-1, \\ 1+\begin{cases} 0 & \text{if } j \neq i_\gamma \\ \tilde{\varphi}_{(\nu)}^{(t)} & \text{if } j = i_\gamma \end{cases} & \text{for } \nu = \gamma, \\ \tilde{\varphi}_{(\nu)}^{(t)} & \text{for } \nu = \gamma+1, \ldots, \hat{s}, \\ \varphi_{(\nu)}^{(t)} & \text{for } \nu = \hat{s}+1, \ldots, \hat{s}, \end{cases} \tag{196}$$

(where the last row is of no interest); in addition, we have the following in the case $j \neq i_\gamma$ /144

$$\varphi_{((\gamma))} = 1, \varphi_{((\gamma))+1} = \varphi_{((\gamma))+2} = \ldots = \varphi_{((\gamma))+j-i_\gamma-1} = 0, \text{ while } \varphi_{((\gamma))+j-i_\gamma} = \tilde{\varphi}_{(\gamma)} + \text{Const.} \quad (j \neq i_\gamma) \tag{197}$$

where the function $\varphi_{((\nu))+j-i_\nu-1}$ occurs only in the case of $j - i_\nu > 1$. Here, we have

$$\tilde{\varphi}_{(\nu)}(t) \not\equiv \text{const.} \quad \text{for } \nu = 1, \ldots, \hat{\varsigma}. \tag{198}$$

[This latter follows from eq.(95) or (97).]

Let us now consider an arbitrary normal solution $x(t)$ of eq.(1), which we will retain in what follows, and let us make statements on the increment of its derivatives $x^{(k)}(t)(k = 0, 1, \ldots, n - 1)$. For this solution $x(t)$, eqs.(19), (31), (26), and (32) are valid. Here, it must be considered that, in eq.(31), the indices ν are arranged in a different manner, namely, first the resonance indices, then the exceptional indices, and finally the principal case indices. In order to retain our above notations, we will replace eq.(31) by

$$\nu x(t) = \sum_{\mu=0}^{\omega_\nu} \frac{t^{\omega_\nu-\mu}}{(\omega_\nu-\mu)!}\, \nu\theta_\mu(t) \quad \text{with} \begin{cases} \omega_\nu = m_\nu, \text{ if } \nu \text{ is the resonance index.} \\ \omega_\nu < m_\nu, \text{ if } \nu \text{ is the exceptional index} \end{cases} \tag{199}$$

Instead of eq.(29), we must then write

$$\nu\theta_0(t) = \nu c_0 \varphi_{((\nu))}(t) \quad \text{with} \quad \nu c_0 \neq 0. \tag{200}$$

Similarly, eq.(19) is rewritten as

$$x(t) = \sum_{\nu=0}^{\hat{s}} \nu x(t). \tag{201}$$

The order of increment of the investigated normal solution $x(t)$ in eq.(32) will be denoted by ω so that, according to eqs.(33) and (34) but using our new notation, the following is valid:

$$\omega = \max_{(\nu=0,1,\ldots,\hat{\varsigma})} \omega_\nu \geq \max_{(\nu \text{ Res})} \omega_\nu = \max_{(\nu \text{ Res})} m_\nu = m. \tag{202}$$

Independent of the investigations, made at the beginning of this Section, on the matrix $\Phi^0(t)$, the following statement can be made directly on the basis of theorem 3. /145

Theorem 20: For a definite normal solution $x(t)$ of eq.(1), relative to

the power order of $x(t)$, $x'(t)$, ..., $x^{(j)}(t)$, ..., $x^{(n-1)}(t)$, we will obtain a broken curve of the following type:

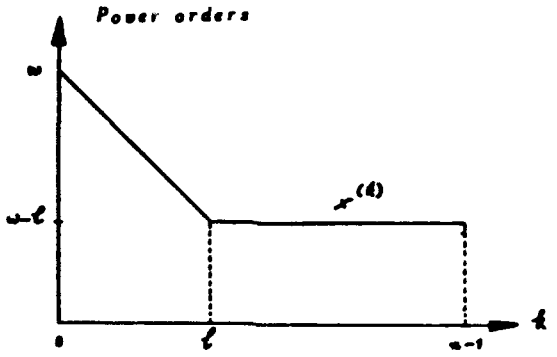

Fig.2

In the case $\ell = 0$, only the horizontal at the height ω remains. [Corresponding patterns are obtained for all ${}^{\nu}x(t)$ at $\nu = 0, 1, \ldots, \hat{\rho}$ from eq.(201) and the pertaining derivatives.]

On the basis of the intuitive statements on the construction of the matrix $\Phi^{0}(t)$, stricter statements can be made. Let j be located in the interval (195). According to eqs.(23) and (25), we have the following for a resonance index or for an exceptional index ν:

$$ {}^{\nu}\theta_{0}(t) = {}^{\nu}d_{0}\, t^{\varrho_{(\nu\nu)}}(t) + \text{Const.} \quad (\nu = 0,1,\ldots,\hat{\rho};\ \nu \neq \varkappa) \tag{203} $$

[see eqs.(30) and (31)]. Consequently, in view of eq.(196), we have:

Theorem 21: For an index $\nu \leq \hat{\rho}$, differing from γ, all ${}^{\nu}x^{(\mu)}(t)$ for $\mu = 0, 1, \ldots, n-1$, have the same power order ω_{ν} (see Fig.3). The same state-

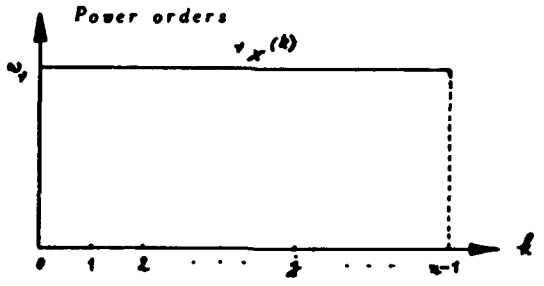

Fig.3

ment applies also to the index $\nu = \gamma$ in the case of $j = i_\gamma$. This theorem /146 has its necessary complement in the following theorem:

Theorem 22: If, in the case $j \neq i_\gamma$ [see eq.(195)], we have $\gamma \leq \hat{\rho}$, the power orders of the derivatives ${}^{\gamma}x(t)$ successively decrease by 1, beginning with ω_γ, down to the order

$$l_\gamma = \text{Min}\,(j - i_\gamma, \omega_\gamma) \tag{204}$$

after which they remain constant and equal to $\omega_\gamma - (j - i_\gamma)$ in the case of $\omega_\gamma \geq j - i_\gamma$ while they will be equal to 0 in the case of $\omega_\gamma \leq j - i_\gamma$ (see Fig.4).

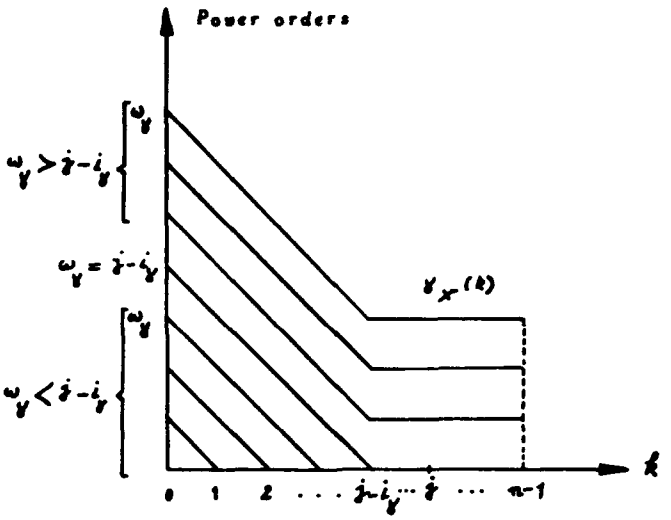

Fig.4

Proof: In eq.(199), according to eqs.(23), (24), (25), and (30) we have /147

$$\begin{cases} {}^{\gamma}\theta_\mu(t) = \sum_{\lambda=0}^{\mu} {}^{\gamma}d_{\mu-\lambda}\, \varphi_{i_\gamma+\lambda}^{(t)} & \text{for } \mu = 0, 1, \ldots, \omega_\gamma - 1, \\ {}^{\gamma}\theta_{\omega_\gamma}(t) = \sum_{\lambda=0}^{\omega_\gamma-1} {}^{\gamma}d_{\omega_\gamma-\lambda}\, \varphi_{i_\gamma+\lambda}^{(t)} + \sum_{\lambda=0}^{m_\gamma-1} r_{i_\gamma+\lambda}^{*(t)} \varphi_{i_\gamma+\lambda}^{(t)}. \end{cases} \tag{205}$$

Because of eq.(197),

$${}^{\gamma}\theta_\mu(t) = \text{Const.} \quad \text{for } \mu = 0, 1, \ldots, j - i_\gamma - 1. \tag{206}$$

In the case of $\omega_\gamma > j - i_\gamma$, the coefficients ${}^{\gamma}\theta_\mu(t)$ of $t^{\omega_\gamma - \mu}$ in eq.(199) are constant for $\mu = 0, 1, \ldots, j - (i_\gamma + 1)$ according to eq.(197), with the

coefficient of t^{ω_γ} being different from zero because of eq.(200). Conversely, the coefficient of $t^{\omega_\gamma-(j-i_\gamma)}$ is not constant. An application of theorem 3 to theorem 20 will demonstrate the correctness of theorem 22 for the case in question here.

In the case $\omega_\gamma \leq j - i_\gamma$, only the powers $t^{\omega_\nu-\mu}$ with $\mu = 0, 1, \ldots, \omega_\gamma \leq$ $\leq j - i_\gamma$ with constant coefficients will occur in eq.(199), where again the coefficient of t^{ω_γ} differs from zero. From this, the statement of theorem 22 follows also for this case.

By means of theorems 21 and 22, the theorem 20 can be made somewhat more rigorous. Let us investigate a certain normal solution $x(t)$ of eq.(1). Since, according to eq.(48), we have $a_{n-j}(t) \not\equiv 0$, all derivatives $x^{(k)}(t)$ with $k = j, j + 1, \ldots, n - 1$ have the same order of magnitude in accordance with theorem 4 (see also the remark made there). Consequently, only the power orders for $k = 0, 1, \ldots, j$ remain to be discussed. Here, several cases must be differentiated.

If we have the following in eq.(202) [see also eq.(195)]

$$\omega > \omega_\gamma \tag{207}$$

or

$$\omega = \omega_\gamma \quad \textit{and simultaneously} \quad j = i_\gamma , \tag{208}$$

then the highest coefficient $\Psi_0(t)$ in eq.(32) is not constant and all derivatives $x^{(k)}(t)(k = 0, 1, \ldots, n - 1)$ have the same power order ω (see theorem 21). Consequently, in Fig.2 we have $\ell = 0$. Figure 5 shows the power orders of the /148 derivatives of $x(t)$ as a dotted line and those of $^{\nu}x(t)$ as a dot-dash line*.

If

$$\omega = \omega_\gamma, \; j \neq i_\gamma, \tag{209}$$

* All quantities $^{\nu}x(t)$, not entered in the illustrations given below, yield horizontals located below the dotted curve.

then three additional cases must be differentiated.

a) An index $\nu = r$ with $r \leq \hat{\delta}$ and $r \neq \gamma$ exists, for which

$$\omega_r = \omega = \omega_\gamma \tag{210}$$

Also in this case, the quantity $\Psi_0(t)$ in eq.(32) is not constant so that we again have $\ell = 0$ in Fig.2. Now, Fig.6, in which $^r x(t)$ with its derivatives is

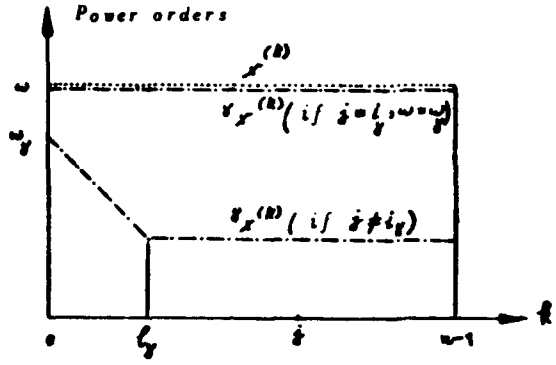

Fig.5

plotted as a dashed line, illustrates the power orders of the derivatives.

b) If, in addition to eq.(209), the following applies for all indices $\nu \leq \hat{\delta}$ differing from γ /149

$$\max_{(\nu=0,\ldots,\hat{\delta};\ \nu\neq\gamma)} \omega_\nu \leq \omega_\gamma - \ell_\gamma < \omega_\gamma \,, \tag{211}$$

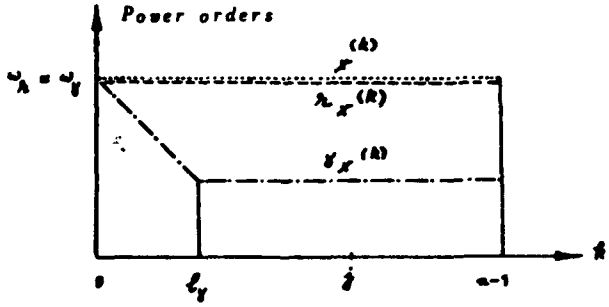

Fig.6

then all $\Psi_\delta(t)$ for $\delta = 0, 1, \ldots, \ell_\gamma - 1$ in eq.(32) are constant, whereas $\Psi_{\ell_\gamma}(t)$ is not constant. Consequently, we have in Fig.2 $\ell = \ell_\gamma = \text{Min}\,(j - i_\gamma, \omega_\gamma)$; see also eq.(204).

c) If at least one index q exists so that, in addition to eq.(209), we also have

$$\omega_\gamma > \max_{(\nu=0,\ldots,\gamma;\ \nu\neq\gamma)} \omega_\nu \qquad \omega_\gamma = \omega_q > \omega_\gamma - \ell_\gamma \tag{212}$$

then $\omega_\gamma - \omega_q$ is the smallest index for which $\Psi_0(t)$ in eq.(32) is not constant.

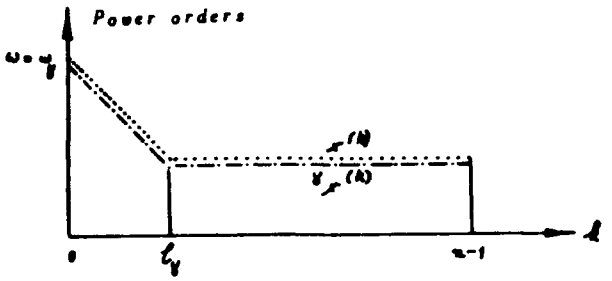

Fig.7

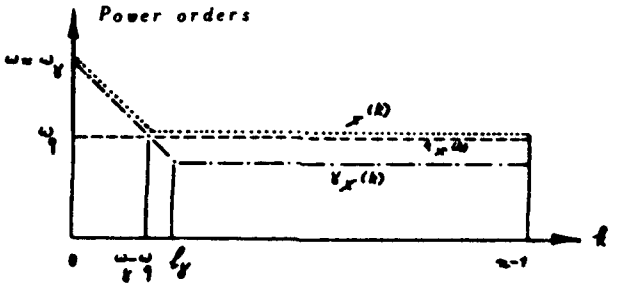

Fig.8

This yields the pattern shown in Fig.8*.

Section 10. Construction of Solutions x(t) of Eq.(1) in which a Given Derivative has the Minimal Order /150

We will now investigate how high the power order of a singular solution of eq.(1) must be in order that, at given k (k = 1, 2, ..., n - 1), the quantity $x^{(k)}(t)$ has a minimal power order.

For this, a complement to theorem 19 is required, whose correctness can be read from the matrix examples of theorem 19 in the same manner as the proof of

* With respect to b) and c) see theorem 24 in Sect.10.

theorem 19 itself. This can be formulated as follows:

Theorem 23: In the matrix $\Phi^0(t)$, we have [see eqs.(160) and (191)]

$$\left\{\begin{array}{l} \text{if} \quad \nu < \varkappa \\ \left(\varphi_{(\varkappa\nu)}, \ldots, \varphi_{[\varkappa\nu]}\right) = \\ = \left(0_1, 0_2, \ldots, 0_{i_{\nu+1}-i_\nu}, \tilde{\varphi}_{(\nu)}, \tilde{\varphi}_{(\nu+1)}, \ldots, \tilde{\varphi}_{[\nu]}\right) - \\ - \left(\tilde{\varphi}_{(\nu\varkappa)}, \tilde{\varphi}_{(\nu\varkappa+1)}, \ldots, \tilde{\varphi}_{(\nu+1)+i_{\nu+1}-i_\nu-1}, \tilde{\varphi}_{(\nu+1)+i_{\nu+1}-i_\nu}, \tilde{\varphi}_{(\nu+1)+i_{\nu+1}-i_\nu+1}, \ldots, \tilde{\varphi}_{(\nu\varkappa+m_\nu-1)}\right), \end{array}\right. \tag{213}$$

$$\left\{\begin{array}{l} \text{if} \quad \nu = \varkappa \text{ and } j \neq i_\varkappa \quad {}^{12)} \\ \left(\varphi_{(\varkappa(j))}, \ldots, \varphi_{[\varkappa\nu]}\right) = \left(1_1, 0_2, \ldots, 0_{j-i_\varkappa}, \tilde{\varphi}_{(\varkappa)}, \tilde{\varphi}_{(\varkappa)+1}, \ldots, \tilde{\varphi}_{[\varkappa]}\right), \end{array}\right. \tag{214*}$$

$$\left\{\begin{array}{l} \text{if} \quad \nu = \varkappa \text{ and } j = i_\varkappa \\ \left(\varphi_{(\varkappa(j))}, \ldots, \varphi_{[\varkappa\nu]}\right) = \left(1 + \tilde{\varphi}_{(\varkappa)}, \tilde{\varphi}_{(\varkappa)+1}, \ldots, \tilde{\varphi}_{[\varkappa]}\right), \end{array}\right. \tag{215}$$

$$\left\{\begin{array}{l} \text{if} \quad \varkappa = \nu < \lambda \\ \left(\varphi_{(\varkappa\nu)}, \ldots, \varphi_{[\varkappa\nu]}\right) = \left(\tilde{\varphi}_{(\nu)}, \tilde{\varphi}_{(\nu)+1}, \ldots, \tilde{\varphi}_{(\nu)+i_\nu-j-1}, 1+\tilde{\varphi}_{(\nu)+i_\nu-j}, \right. \\ \qquad\qquad \left. \tilde{\varphi}_{(\nu)+i_\nu-j+1}, \ldots, \tilde{\varphi}_{[\nu]}\right), \end{array}\right. \tag{216}$$

$$\left\{\begin{array}{l} \text{if} \quad \nu > \lambda \\ \left(\varphi_{(\varkappa\nu)}, \ldots, \varphi_{[\varkappa\nu]}\right) = \left(\varphi_{(\nu)}, \ldots, \varphi_{[\nu]}\right). \end{array}\right. \tag{217}$$

In the cases (215), (216), and (217), we have $[[\nu]] = [\nu]$. For producing the general proof, it should be stated that the zeros in eqs.(213) and (214) are due to the permutation matrices $\mathfrak{P}_\mu$ while the negative terms in eq.(213) are due to the superposition matrices $\mathfrak{P}_{\ell(\mu)}$. The matrices $\mathfrak{S}_\mu$* have no effect at all on

* In the case $\nu = \varkappa = 0$, only $(\varphi_{((0))}, \varphi_{((0))+1}, \ldots, \varphi_{((0))+j-1}) = (1_1, 0_2, \ldots, 0_j)$ remains of eq.(214) which means that, in the case $j = 1$, we only have $(\varphi_{((0))}) = (1_1)$. The situation is similar for eq.(213).

the matrices ${}^{\nu}\tilde{\Phi}_{\mu}$.

First, let us repeat Fig.1 with respect to the minimal order of the derivatives ${}^{\nu}x^{(k)}(t)$ [see eq.(19)], with k as the abscissa.

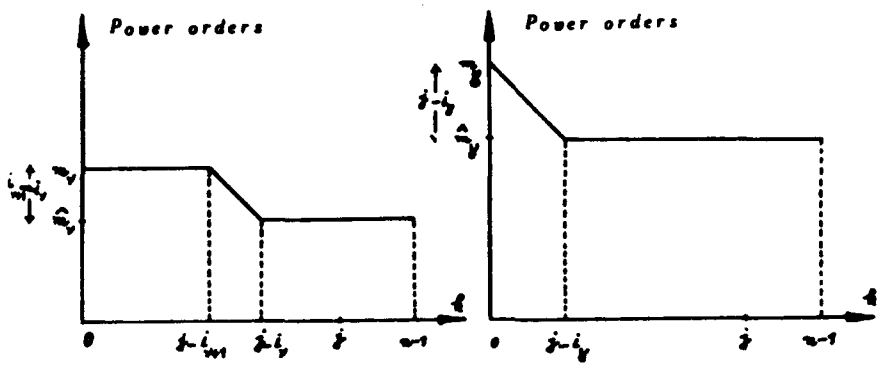

Fig.9(a) Fig.9(b)

With a minor modification of eq.(186), we can then define:

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$$m_{\underline{\nu}} = \left(\begin{smallmatrix}\text{Max}\\ \nu=0,\dots,\gamma\\ \nu\ \text{Res.}\end{smallmatrix}\right) m_{\nu} \quad \text{and} \quad m_{\bar{\underline{\nu}}} = \left(\begin{smallmatrix}\text{Max}\\ \nu=\gamma+1,\dots,\lambda,\dots,\hat{s}\\ \nu\ \text{Res.}\end{smallmatrix}\right) m_{\nu}, \tag{218}$$

where γ is defined by eq.(195). Then, the relations (185) apply, provided that λ is substituted by γ (see footnote on p.109); this had been taken into consideration already in Fig.9. If we have

$$n = \text{Max}\,(m_{\underline{\nu}},\ m_{\bar{\underline{\nu}}}) = m_{\bar{\underline{\nu}}}, \tag{219}$$

then, according to the relations (185) modified in this manner, the minimal orders of all derivatives $x^{(k)}(t)$ ($k = 0, 1, \dots, n-1$) of a normal solution $x(t)$ of eq.(1) are constant and equal to $m_{\underline{\nu}} = \hat{m}_{\underline{\nu}}$. This constitutes no problem.

Then, only the case

$$n = \text{Max}\,(m_{\underline{\nu}},\ m_{\bar{\underline{\nu}}}) = m_{\underline{\nu}} \tag{220}$$

remains to be considered. If γ is the resonance index, i.e., if $\gamma = \bar{\bar{\nu}}$, the power order of the derivative $x^{(k)}(t)$ of a normal solution is equal to the power order m_{γ} of ${}^{\gamma}x^{(k)}(t)$ according to Fig.9b. Accordingly, the only case of interest

is that in which γ represents the exceptional index, i.e.,

$$\bar{\bar{\nu}} < \gamma . \tag{221}$$

If $k \leq j - i_{\bar{\bar{\nu}}} + 1$, then ${}^{\bar{\bar{\nu}}}x^{(k)}(t)$ will yield the minimal power order of a normal solution $x(t)$ in accordance with Fig.9a. Information on the case

$$k > j - i_{\bar{\bar{\nu}}+1} \tag{222}$$

is obtained by the following theorem:

Theorem 24: If a given normal solution $x(t)$ of eq.(1) is resolved, under the assumptions of eqs.(195), (220), (221), (222) and in view of eq.(201), in the following form

$$x(t) = x_{*}(t) + x_{**}(t) \tag{223}$$

with

$$x_{*}(t) = \sum_{\nu=\bar{\bar{\nu}}}^{\gamma} {}^{\nu}x(t) = \sum_{\delta=0}^{\omega} \frac{t^{\omega-\delta}}{(\omega-\delta)!} \mathfrak{x}_{\delta}(t) \tag{224}$$

[see eq.(32)], in which case

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$$\omega = \omega_{\gamma} = \hat{m}_{\bar{\bar{\nu}}}\, t\left(j - i_{\bar{\bar{\nu}}}\right) = m_{\bar{\bar{\nu}}}\, t\left(j - i_{\bar{\bar{\nu}}+1}\right) \tag{225}$$

[see eq.(150)] can be selected, then the constants ${}^{\nu}\alpha_{\mu-\gamma}$ occurring in the ${}^{\nu}x(t)$ of eq.(224) according to eq.(26) can be so modified that $\Psi_0(t)$, $\Psi_1(t)$, ..., $\Psi_{l-1}(t)$ are constant while $\Psi_l(t)$ is not constant, with

$$l = j - i_{\bar{\bar{\nu}}}. \tag{226}$$

On substituting this modified function $x_{*}(t)$ in eq.(223), the quantity $x_{**}(t)$ can be so selected that the derivatives $x^{(k)}(t)$ of $\omega = \omega_{\gamma}$ from eq.(225), for $k = 0$, will decrease with increasing k by 1 each time until the power order

$$M = \mathrm{Max}\left(\hat{m}_{\bar{\bar{\nu}}}, \hat{m}_{\bar{\bar{\mu}}}\right). \tag{227}$$

is reached. In the case $M = \hat{m}_{\bar{\bar{\nu}}}$, the decrease of the power order proceeds to the l^{th} derivative with l according to eq.(226) while, in the case $M = m_{\bar{\bar{\nu}}} = \hat{m}_{\bar{\bar{\nu}}}$ it proceeds to the l^{*th} derivative with

$$l^* = l - (m_{\underline{\nu}} - m_{\underline{\nu}}) \tag{228}$$

Figures 10 and 11, in which x(t) from eq.(223) is shown as a dotted line, illustrate the conditions stipulated in theorem 24: Fig.10 for the case l of eq.(226) and Fig.11 for the case l^* of eq.(228).

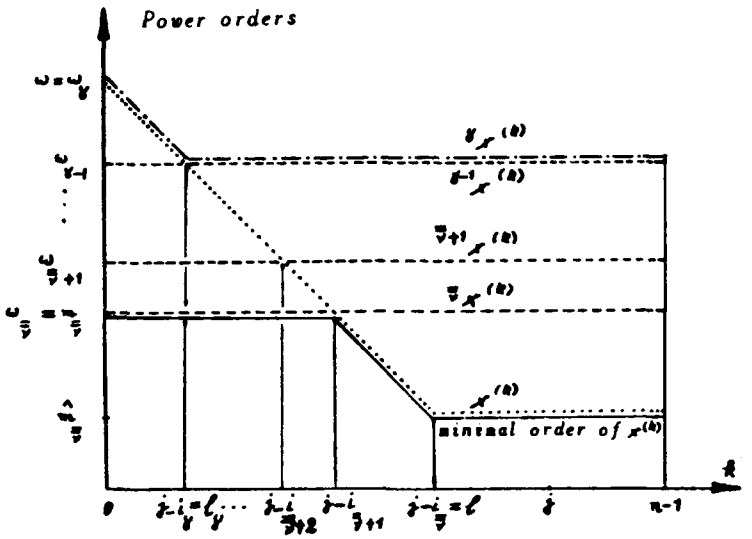

Fig.10

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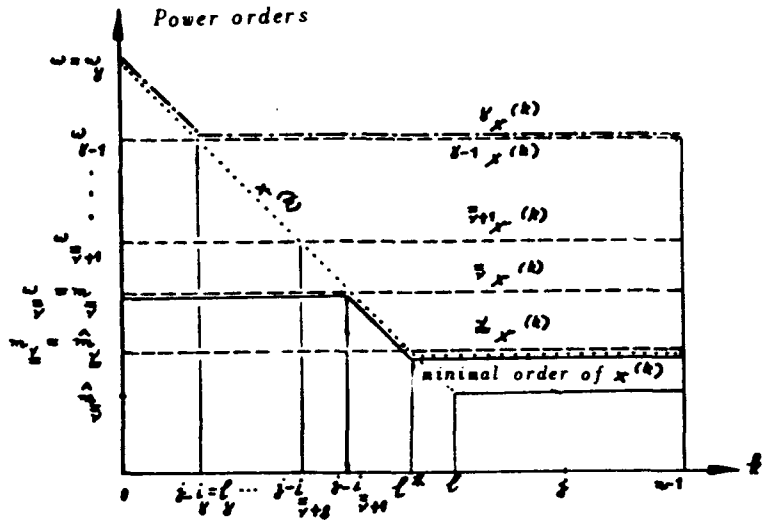

Fig.11

Proof: First, it should be recalled that the summation indices $\bar{\bar{\nu}} + 1$, $\bar{\bar{\nu}} + 2$, ..., γ which occur in eq.(224) in addition to $\bar{\bar{\nu}}$ must be exceptional in-

dices, which means that, for these indices, the constants ${}^{\nu}d_{\mu-\gamma}$ in a normal solution $x(t)$ can be arbitrarily selected. Consequently, because of [see eqs.(74) and (150)]

$$\omega = \omega_{\gamma} = \hat{m}_{\bar{\nu}} + (\gamma - i_{\bar{\nu}}) < \hat{m}_{\gamma} + \gamma - i_{\gamma} = m_{\gamma}$$

we can select eq.(224) as the upper limit of the sum. If, analogously to eq.(225), we assume for $\nu = \bar{\bar{\nu}} + 1, \bar{\bar{\nu}} + 2, \ldots, \gamma - 1$,

$$\omega_{\nu} = \hat{m}_{\bar{\nu}} + (i_{\nu+1} - i_{\bar{\nu}}) = m_{\bar{\nu}} + (i_{\nu+1} - i_{\bar{\nu}+1}) < m_{\nu} \tag{229}$$

and take into consideration [see eq.(150)]

$$\omega_{\bar{\nu}} = m_{\bar{\nu}} = \hat{m}_{\bar{\nu}} + (i_{\bar{\nu}+1} - i_{\bar{\nu}}), \tag{230}$$

then eq.(224) can be written in the form of

$$x_{*}(t) = \sum_{\nu=\bar{\bar{\nu}}}^{\gamma} \left(\sum_{\mu=0}^{\omega_{\nu}} \frac{t^{\omega_{\nu}-\mu}}{(\omega_{\nu}-\mu)!} \, {}^{\nu}\theta_{\mu}(t) \right) \tag{231}$$

[see eq.(199) or (31)] with [see eqs.(26) or (24)]

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$$\begin{cases} {}^{\nu}\theta_{\mu}(t) = \sum_{\lambda=0}^{\mu} {}^{\nu}d_{\mu-\lambda} \, \varphi_{((\nu))+\lambda}(t) \quad (\mu = 0, 1, \ldots, \omega_{\nu}-1), \\ {}^{\nu}\theta_{\omega_{\nu}}(t) = \sum_{\lambda=0}^{\omega_{\nu}-1} {}^{\nu}d_{\omega_{\nu}-\lambda} \, \varphi_{((\nu))+\lambda}(t) + \sum_{\lambda=0}^{m_{\nu}-1} v^{*}_{((\nu))+\lambda}(t) \varphi_{((\nu))+\lambda}(t), \end{cases} \tag{232}$$

where the $v^{*}_{((\nu))}(t)$ periodic with P and having a mean value zero can be selected [see (Bibl.1) eq.(103)]. A comparison of the t-powers in eqs.(224) and (231), using the notations

$$\mu_{\nu\delta} = \delta - (\gamma - i_{\nu+1}) \quad \text{for} \quad \begin{cases} \nu = \bar{\nu}, \bar{\nu}+1, \ldots, \gamma-1 \\ \delta \geq \gamma - i_{\nu+1} \end{cases} \tag{233}$$

will yield

$$\chi_{\delta}(t) = {}^{\gamma}\theta_{\delta} + \sum_{\nu=\bar{\nu}}^{\gamma-1} {}^{\nu}\theta_{\mu_{\nu\delta}}(t), \quad \delta = 0, 1, \ldots, \gamma - i_{\bar{\nu}} = \ell, \ldots, \omega_{\gamma}, \tag{234}$$

which can be used only up to $\delta = \ell$, in which case any summands with $\mu_{\nu\delta}$ not defined in eq.(233) must be omitted. It must be noted here that the quantities (225) and (229) are the highest occurring exponents of t at ${}^{\gamma}\theta_{0}(t)$ or ${}^{\nu}\theta_{0}(t)$.

In addition to eq.(206), we will also need the following relation which has been obtained by means of eq.(214) from eq.(232):

$$
{}^{\nu}\theta_{\delta}^{(t)} = \sum_{\lambda=0}^{\mu_{\nu-1,\delta}} {}^{\nu}d_{\mu_{\nu-1,\delta}-\lambda}\,\tilde{\varphi}_{(\nu)+\lambda}^{(t)} + {}^{\nu}d_{\delta}
$$

$$
\text{with} \quad \delta = j - i_{\nu}, \ldots, \begin{cases} \ell, & \text{if } \hat{m}_{\nu} > 0 \\ \ell - 1, & \text{if } \hat{m}_{\nu} = 0 \end{cases} \tag{235}
$$

In the case $m_{\nu} = 0$, the second sum of the second formula in the system (232) will also occur in eq.(235) for $\delta = 1$, which must be specifically taken into consideration later for eqs.(237), (241), (243), and (244). For $\delta = 0, 1, \ldots, j - i_{\nu} - 1$, eq.(206) is contained in eq.(235) since in that case, according to eq.(235), the quantity $\mu_{\nu-1,\,\delta} - r$ becomes negative, meaning that the sum does not occur in eq.(235). This formula is applicable no matter whether $j = i_{\nu}$ or $j \neq i_{\nu}$, as can be confirmed by means of eqs.(214) and (215). /156

In accordance with eq.(232), we obtain for the remaining summands in eq.(234), i.e., for

$$
\nu = \bar{\nu}, \bar{\nu}+1, \ldots, \varkappa - 1, \tag{236}
$$

by means of eq.(213),

$$
\begin{aligned}
{}^{\nu}\theta_{\mu_{\nu\delta}}^{(t)} = &- \sum_{\lambda=0}^{\mu_{\nu\delta}} {}^{\nu}d_{\mu_{\nu\delta}-\lambda}\,\tilde{\varphi}_{(\nu+1)+\lambda}^{(t)} + \\
&+ \sum_{\lambda=i_{\nu+1}-i_{\nu}}^{\mu_{\nu\delta}} {}^{\nu}d_{\mu_{\nu\delta}-\lambda}\,\tilde{\varphi}_{(\nu)+\lambda-(i_{\nu+1}-i_{\nu})}^{(t)} = \\
= &- \sum_{\lambda=0}^{\mu_{\nu\delta}} {}^{\nu}d_{\mu_{\nu\delta}-\lambda}\,\tilde{\varphi}_{(\nu+1)+\lambda}^{(t)} + \sum_{\lambda=0}^{\mu_{\nu-1,\delta}} {}^{\nu}d_{\mu_{\nu-1,\delta}-\lambda}\,\tilde{\varphi}_{(\nu)+\lambda}^{(t)}.
\end{aligned} \tag{237}
$$

Since, according to eq.(233), we have

$$
\delta \gtrless j - i_{\nu+1} \gtrless j - i_{\nu} \tag{238}
$$

the sum does not occur in eq.(234) for the case that

$$
\delta = 0, 1, \ldots, j - i_{\nu} - 1, \tag{239}
$$

so that here $\Psi_\delta(t)$ according to eq.(235) [resp. eq.(206)] is constant*. Consequently, we now need consider only

$$\delta = j - i_\gamma,\ j - i_\gamma + 1, \ldots, 1 \tag{240}$$

Next, we calculate, in accordance with eqs.(234), (235), and (237),

$$\begin{aligned}\chi_\delta^{(t)} = &\sum_{\lambda=0}^{\mu_{\gamma-1,\delta}} {}^{\nu}d_{\mu_{\gamma-1,\delta}-\lambda}\,\tilde{\varphi}_{(\gamma)+\lambda}^{(t)} + {}^{\nu}d_\delta\, t \\ &+ \sum_{\nu=\bar{\bar{\gamma}}}^{\nu-1}\left(-\sum_{\lambda=0}^{\mu_{\nu\delta}} {}^{\nu}d_{\mu_{\nu\delta}-\lambda}\,\tilde{\varphi}_{(\nu+1)+\lambda}^{(t)} + \sum_{\lambda=0}^{\mu_{\nu-1,\delta}} {}^{\nu}d_{\mu_{\nu-1,\delta}-\lambda}\,\tilde{\varphi}_{(\nu)+\lambda}^{(t)}\right) \\ = &\,{}^{\nu}d_\delta + \sum_{\nu=\bar{\bar{\gamma}}}^{\nu-1}\sum_{\lambda=0}^{\mu_{\nu\delta}}\left({}^{\nu+1}d_{\mu_{\nu\delta}-\lambda} - {}^{\nu}d_{\mu_{\nu\delta}-\lambda}\right)\tilde{\varphi}_{(\nu+1)+\lambda}^{(t)} + \\ &+ \sum_{\lambda=0}^{\mu_{\bar{\bar{\gamma}}-1,\delta}} {}^{\bar{\bar{\gamma}}}d_{\mu_{\bar{\bar{\gamma}}-1,\delta}-\lambda}\,\tilde{\varphi}_{(\bar{\bar{\gamma}})+\lambda}^{(t)}.\end{aligned} \tag{241}$$

Then, we select the arbitrary ${}^{\nu}d_\tau$ such that the parentheses vanish, i.e., /157

$${}^{\nu+1}d_{\mu_{\nu\delta}-\lambda} = {}^{\nu}d_{\mu_{\nu\delta}-\lambda} \qquad \nu = \bar{\bar{\nu}}, \bar{\bar{\nu}}+1, \ldots, \gamma-1, \tag{242}$$

where only eq.(27) for $\nu = \bar{\bar{\nu}}$ must be taken into consideration. Then, the only remaining terms of eq.(241) are

$$\chi_\delta^{(t)} = {}^{\gamma}d_\delta + \sum_{\lambda=0}^{\mu_{\bar{\bar{\gamma}}-1,\delta}} {}^{\bar{\bar{\gamma}}}d_{\mu_{\bar{\bar{\gamma}}-1,\delta}-\lambda}\,\tilde{\varphi}_{(\bar{\bar{\gamma}})+\lambda}^{(t)}. \tag{243}$$

Here, the sum occurs only if [see eq.(231)]

$$\mu_{\bar{\bar{\gamma}}-1,\delta} = \delta - (j - i_{\bar{\bar{\gamma}}}) \geqq 0$$

i.e., because of eq.(226), if

$$\delta \geq \ell$$

Consequently, according to eq.(241), the quantities $\Psi_0, \Psi_1, \ldots, \Psi_{\ell-1}$ are constant while

$$\chi_\ell^{(t)} = {}^{\gamma}d_\ell + {}^{\bar{\bar{\gamma}}}d_0\,\tilde{\varphi}_{(\bar{\bar{\gamma}})}^{(t)} \tag{244}$$

is not constant [see eq.(198)]. In view of theorem 3 (or theorem 20) this

* At $j = i_\gamma$, these terms do not occur.

proves theorem 24 under the additional stipulation that $x_{**}(t)$ in eq.(223) contains no t-powers higher than t^M, with M from eq.(227). For this, it must first be established that no ν of the series $\bar{\bar{\nu}} + 1, \ldots, \lambda$ can be a resonance index and thus need not furnish any t-power. For such a ν, according to eqs.(74) and (150), we would have

$$\mu_\nu > \hat{\mu}_\nu > \hat{\mu}_{\bar{\nu}} + (i_\nu - i_{\bar{\nu}}) \geq \hat{\mu}_{\bar{\nu}} + (i_{\bar{\nu}+1} - i_{\bar{\nu}}) = \mu_{\bar{\nu}},$$

which constitutes a contradiction to eq.(220). For the resonance indices of the series $\lambda + 1, \ldots, \hat{\rho}$, according to the definition in Section 4, the highest index has the greatest $m_\nu = \hat{m}_\nu$ which means that, according to eq.(220), we must have $m_\nu < m_{\bar{\nu}}$. Here, it should be noted that the case $m = m_{\bar{\nu}} = m_{\underline{\nu}}$ in eq.(220) had been taken care of already when treating eq.(219). This will immediately produce the bound (227) since the indices $\nu > \hat{\rho}$ furnish only periodic components to a normal solution x(t) of eq.(1).

As an application to theorem 24, we can use the example on p.109 with <u>/158</u> $j = \gamma$, in which case we can assume $\nu = 0, 2, 4$ as resonance indices. We here have

$$\gamma = 3,\ i_\gamma = 5,\ \bar{\gamma} = 2,\ m_{\bar{\gamma}} = 7,\ \hat{m}_{\bar{\gamma}} = 5,\ i_{\bar{\gamma}} = 3;$$

so that, according to eq.(225), it can be computed that

$$\omega = \omega_3 = 5 + (8 - 3) = 10$$

and, according to eq.(226)

$$1 = 8 - 3 = 5.$$

In Fig.12, the dotted line indicates the power order of the k^{th} derivative of the solution x(t) according to theorem 24, while the solid line shows the minimal power order according to Fig.9a; the dot-dash line gives $^2x(t)$. This latter line generally also is valid for the solution (223) with ω from eq.(225), i.e., for $\omega = 10$, and can drop below the value $m_2 = 7$ only at a special selection of the constants $^\nu d_\tau$; for the selection (242), this can drop to the minimal

order of $\hat{m}_6 = 5$.

So as to obtain an example also for the case (228), our above statements are modified such that we lower $m_6 = 6$ and add $\nu = 5$ as a resonance index to

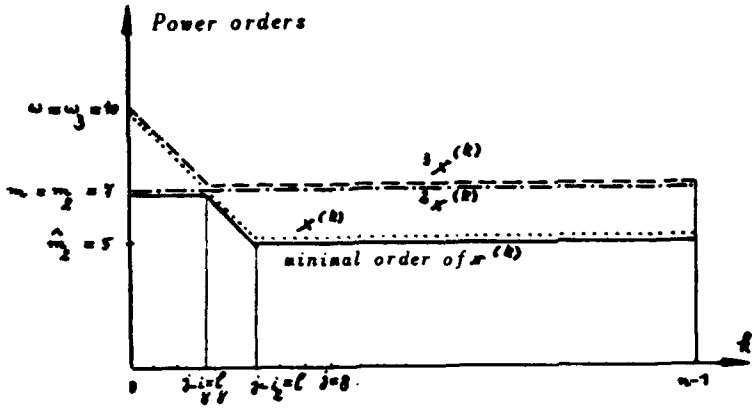

Fig.12

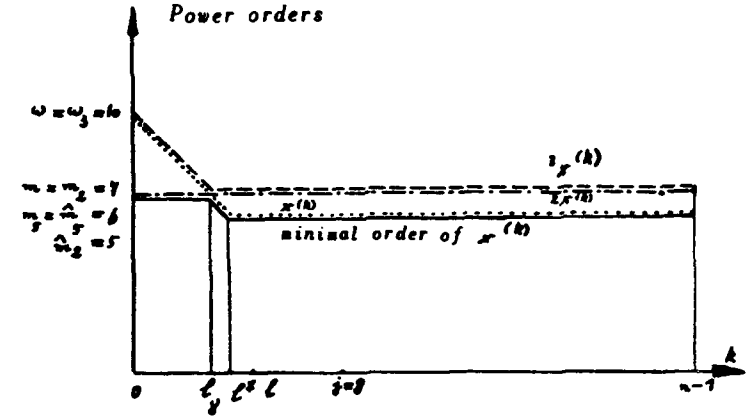

Fig.13

$\nu = 0$, 2, and 4. We now have again /159

$$\omega = \omega_3 = 10;\ 1 = 5,$$

$$\tilde{\nu} = 2,\ m_{\tilde{\nu}} = 7,\ \hat{m}_{\tilde{\nu}} = 5,\ \text{whereas}\ m_{\nu} = \hat{m}_{\nu} = m_5 = \hat{m}_5 = 6\ .$$

Consequently, we have in eq.(227),

$$M = \text{Max}\ (5;6) = 6$$

and, according to eq.(228),

$$1^{*} = 5 - (6 - 5) = 4$$

The new conditions are illustrated in Fig.13.

Section 11. Method for the Formation of Examples /160

Theorem 25: The functions

$$y_\mu(t) = e^{\alpha_\mu t}\,\varphi_\mu(t) \quad \text{for } \mu = 1,\ldots,n, \tag{245}$$

where the $\varphi_\mu(t)$ are n-times continuously differentiable and periodic with the period P, form a fundamental system of a differential equation of the n^{th} order having coefficients periodic with P, provided that the Wronski determinant of the functions $y_1, \ldots, y_n$ differs from zero for each value of t within the interval $0 \le t \le P$. Here, the matrix $\mathfrak{B} = \mathfrak{y}^{-1}(0)\mathfrak{y}(P)$ contains only elementary components of the order 1.

Proof: Obviously, the functions $y_\mu(t)$ satisfy the differential equation of the n^{th} order:

$$\begin{vmatrix} y, & e^{\alpha_1 t}\varphi_1, & \cdots, & e^{\alpha_n t}\varphi_n \\ y', & e^{\alpha_1 t}(\alpha_1\varphi_1 + \varphi_1'), & \cdots, & e^{\alpha_n t}(\alpha_n\varphi_n + \varphi_n') \\ y'', & e^{\alpha_1 t}(\alpha_1^2\varphi_1 + 2\alpha_1\varphi_1' + \varphi_1''), & \cdots, & e^{\alpha_n t}(\alpha_n^2\varphi_n + 2\alpha_n\varphi_n' + \varphi_n'') \\ \vdots & & & \vdots \\ y^{(n)}, & e^{\alpha_1 t}\left(\alpha_1^n\varphi_1 + \binom{n}{1}\alpha_1^{n-1}\varphi_1' + \cdots + \varphi_1^{(n)}\right), & \cdots, & e^{\alpha_n t}\left(\alpha_n^n\varphi_n + \binom{n}{1}\alpha_n^{n-1}\varphi_n' + \cdots + \varphi_n^{(n)}\right) \end{vmatrix} = 0. \tag{246}$$

If the exponential functions are canceled from the columns, a differential equation with coefficients periodic with P is obtained, in which case the factor of the highest derivative which, except for the sign, coincides with the Wronski determinant of the functions $y_1, \ldots, y_n$, according to definition is a continuous function differing from zero. In addition, the following is valid for the fundamental system:

$$\mathfrak{y}(t) = \begin{bmatrix} y_1, & y_2, & \cdots, & y_n \\ y_1', & y_2', & \cdots, & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{bmatrix} = \tag{247}$$

$$= \begin{bmatrix} \alpha_1 \varphi_1 + \varphi_1' & \cdots & \alpha_m \varphi_m + \varphi_m' \\ \vdots & & \vdots \\ \alpha_1^{m-1} \varphi_1 + \binom{m-1}{1} \alpha_1^{m-2} \varphi_1' + \cdots + \varphi_1^{(m-1)} & \cdots & \alpha_m^{m-1} \varphi_m + \binom{m-1}{1} \alpha_m^{m-2} \varphi_m' + \cdots + \varphi_m^{(m-1)} \end{bmatrix} \begin{bmatrix} e^{\alpha_1 t} & & \\ & e^{\alpha_2 t} & \\ & & \ddots \\ & & & e^{\alpha_m t} \end{bmatrix}, \tag{247}$$

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$$\mathfrak{y}(t+P) = \mathfrak{y}(t) \begin{bmatrix} e^{\alpha_1 P} & & & \\ & e^{\alpha_2 P} & & \\ & & \ddots & \\ & & & e^{\alpha_m P} \end{bmatrix} \tag{248}$$

From this, it follows that [see (Bibl.1), eq.(10)]

$$\mathfrak{P} = \begin{bmatrix} e^{\alpha_1 P} & & & \\ & e^{\alpha_2 P} & & \\ & & \ddots & \\ & & & e^{\alpha_m P} \end{bmatrix}. \tag{249}$$

Consequently, the matrix $\mathfrak{P}$ consists of elementary components of the order 1.

Theorem 26: The functions

$$\begin{cases} y_1(t) = e^{\alpha t} \varphi_1(t) \\ y_2(t) = e^{\alpha t} (t \varphi_1(t) + \varphi_2(t)) \\ y_3(t) = e^{\alpha t} \left(\frac{t^2}{2!} \varphi_1(t) + t \varphi_2(t) + \varphi_3(t) \right) \\ \vdots \\ y_m(t) = e^{\alpha t} \left(\frac{t^{m-1}}{(m-1)!} \varphi_1(t) + \frac{t^{m-2}}{(m-2)!} \varphi_2(t) + \cdots + \varphi_m(t) \right), \end{cases} \tag{250}$$

where the $\varphi_1(t)$, ..., $\varphi_m(t)$, are m-times continuously differentiable functions with the period P, form a fundamental system of a differential equation of the m^{th} order having coefficients periodic with P, provided that the Wronski determinant of the functions $y_1(t)$, ..., $y_m(t)$ differs from zero for each value of t within the interval $0 \leq t \leq P$. The corresponding matrix $\mathfrak{P}$ consists of exactly one elementary component of the order m.

Proof: The differential equation of the m^{th} order, which is analogous to eq.(246), after reducing by $e^{\alpha t}$ and introducing the operator

$$D = \frac{d}{dt} \tag{251}$$

as well as the notation

$$\vartheta_{\mu,\lambda}(t)=\sum_{\ell=1}^{\mu}\binom{\mu-1}{\ell-1}(D+a)^{\mu-\ell}\varphi_{\lambda-\ell+1}(t)\ \text{for}\begin{cases}\mu=1,2,\ldots,m+1,\\ \lambda=1,\ldots,m\end{cases}\tag{252}$$

- summands in which the index $\varphi \leq 0$ must be replaced by 0 - can be written in the following form:

$$\begin{vmatrix}
y, & \vartheta_{11}, & t\vartheta_{11}+\vartheta_{12}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{11}+\frac{t^{m-2}}{(m-2)!}\vartheta_{12}+\ldots+\vartheta_{1m}\\
y', & \vartheta_{21}, & t\vartheta_{21}+\vartheta_{22}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{21}+\frac{t^{m-2}}{(m-2)!}\vartheta_{22}+\ldots+\vartheta_{2m}\\
y'', & \vartheta_{31}, & t\vartheta_{31}+\vartheta_{32}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{31}+\frac{t^{m-2}}{(m-2)!}\vartheta_{32}+\ldots+\vartheta_{3m}\\
\vdots & & \vdots & & \\
y^{(m-1)}, & \vartheta_{m1}, & t\vartheta_{m1}+\vartheta_{m2}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{m1}+\frac{t^{m-2}}{(m-2)!}\vartheta_{m2}+\ldots+\vartheta_{mm}\\
y^{(m)}, & \vartheta_{m+1,1}, & t\vartheta_{m+1,1}+\vartheta_{m+1,2}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{m+1,1}+\frac{t^{m-2}}{(m-2)!}\vartheta_{m+1,2}+\ldots+\vartheta_{m+1,m}
\end{vmatrix}=0\tag{253}$$

The first row is obtained directly, while the other rows are obtained by complete induction with reference to the row index. If the power factors are eliminated by forming column combinations, a differential equation having coefficients periodic with P will be obtained. Again, the factor of the highest derivative $y^{(n)}(t)$, except for the sign, is equal to the Wronski determinant of the functions $y_1(t), \ldots, y_n(t)$, i.e., is continuous and different from zero. In addition, analogous to eq.(247), we have

$$\eta(t)=\begin{bmatrix}y_1, & y_2, & \ldots, & y_m\\ y_1', & y_2', & \ldots, & y_m'\\ y_1'', & y_2'', & \ldots, & y_m''\\ \vdots & & & \vdots\\ y_1^{(m-1)}, & y_2^{(m-1)}, & \ldots, & y_m^{(m-1)}\end{bmatrix}=\tag{254}$$

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$$=e^{mat}\cdot\begin{bmatrix}
\vartheta_{11}, & t\vartheta_{11}+\vartheta_{12}, & \frac{t^2}{2!}\vartheta_{11}+t\vartheta_{12}+\vartheta_{13}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{11}+\ldots+\vartheta_{1m}\\
\vartheta_{21}, & t\vartheta_{21}+\vartheta_{22}, & \frac{t^2}{2!}\vartheta_{21}+t\vartheta_{22}+\vartheta_{23}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{21}+\ldots+\vartheta_{2m}\\
\vartheta_{31}, & t\vartheta_{31}+\vartheta_{32}, & \frac{t^2}{2!}\vartheta_{31}+t\vartheta_{32}+\vartheta_{33}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{31}+\ldots+\vartheta_{3m}\\
\vdots & & & & \vdots\\
\vartheta_{m1}, & t\vartheta_{m1}+\vartheta_{m2}, & \frac{t^2}{2!}\vartheta_{m1}+t\vartheta_{m2}+\vartheta_{m3}, & \ldots, & \frac{t^{m-1}}{(m-1)!}\vartheta_{m1}+\ldots+\vartheta_{mm}
\end{bmatrix},$$

$$\eta(t+\tau)=e^{-\alpha(t+\tau)}\begin{bmatrix} y_{11}, (t+\tau)y_{11}+y_{12}, \dots, \frac{(t+\tau)^{m-1}}{(m-1)!}y_{11}+\dots+y_{1m} \\ y_{21}, (t+\tau)y_{21}+y_{22}, \dots, \frac{(t+\tau)^{m-1}}{(m-1)!}y_{21}+\dots+y_{2m} \\ y_{31}, (t+\tau)y_{31}+y_{32}, \dots, \frac{(t+\tau)^{m-1}}{(m-1)!}y_{31}+\dots+y_{3m} \\ \vdots \\ y_{m,1}, (t+\tau)y_{m,1}+y_{m,2}, \dots, \frac{(t+\tau)^{m-1}}{(m-1)!}y_{m,1}+\dots+y_{m,m} \end{bmatrix} = \tag{255}$$

$$=\eta(t)\cdot e^{-\alpha\tau}\cdot\begin{bmatrix} 1 & \tau & \frac{\tau^2}{2!} & \dots & \frac{\tau^{m-1}}{(m-1)!} \\ & 1 & \tau & \dots & \frac{\tau^{m-2}}{(m-2)!} \\ & & \ddots & & \vdots \\ & & & & 1 \end{bmatrix}.$$

From this, again in accordance with another paper [(Bibl.1), eq.(10)], it follows that

$$\mathfrak{P}=e^{-\alpha\tau}\begin{bmatrix} 1 & \tau & \frac{\tau^2}{2!} & \dots & \frac{\tau^{m-1}}{(m-1)!} \\ & 1 & \tau & \dots & \frac{\tau^{m-2}}{(m-2)!} \\ & & \ddots & & \vdots \\ & & & & 1 \end{bmatrix} = \tag{256}$$

$$=\begin{bmatrix} 1 & \tau & \frac{\tau^2}{2!} & \dots & \frac{\tau^{m-1}}{(m-1)!} \\ & 1 & \tau & \dots & \frac{\tau^{m-2}}{(m-2)!} \\ & & \ddots & & \vdots \\ & & & & 1 \end{bmatrix}\begin{bmatrix} e^{\alpha\tau} & & & \\ & e^{\alpha\tau} & & \\ & & \ddots & \\ & & & e^{\alpha\tau} \end{bmatrix}.$$

Consequently, the matrix $\mathfrak{P}$ consists of a single elementary component of the /164 order m. Theorem 25 for $\mu = 1$ represents a special case of theorem 26 for $m = 1$.

From this, we directly obtain the following theorem:

Theorem 27: If, in forming the differential equation (246) or eq.(253) from several function systems of the type of eq.(250), i.e., for example,

$$y_{(1)}, y_{(1)+1}, \dots, y_{(1)+m_1-1}; y_{(2)}, y_{(2)+1}, \dots, y_{(2)+m_2-1}; \dots; y_{(s)}, y_{(s)+1}, \dots y_{(s)+m_s-1},$$

a differential equation of the order

$$n = m_1 + m_2 + \dots + m_s$$

is obtained, with continuous coefficients P, and one matrix

$$\mathfrak{P} = \begin{bmatrix} \mathfrak{P}_1 & & & \\ & \mathfrak{P}_2 & & \\ & & \ddots & \\ & & & \mathfrak{P}_s \end{bmatrix}, \tag{257}$$

where $\mathfrak{P}_\nu$ is of the order m_ν.

Section 12. Examples /165

Example 1: In accordance with theorem 26, with $m = 2$ and $\alpha = 0$, we start from the fundamental system

$$\hat{y}_{(1)} = \sin t, \quad \hat{y}_{[1]} = t \cdot \sin t - 3\cos t + 1. \tag{258}$$

These functions are solutions of the differential equation [see eq.(253) or the formula analogous to eq.(246)]

$$\hat{L}[\hat{y}] \equiv \hat{y}'' - \frac{2\sin t\cos t + \sin t}{3 + \sin^2 t - \cos t}\hat{y}' + \frac{4 + \cos^2 t}{3 + \sin^2 t - \cos t}\hat{y} = 0, \tag{259}$$

in which the denominator is positive. The corresponding solution matrix $\hat{\mathfrak{y}}(t)$ reads

$$\hat{\mathfrak{y}}(t) = \hat{\Phi}(t) \cdot e^{\hat{\mathfrak{A}}t} = \left[\begin{array}{c|c} \sin t & 1-3\cos t \\ \hline \cos t & 4\sin t \end{array}\right] \left[\begin{array}{c|c} 1 & t \\ \hline 0 & 1 \end{array}\right] \tag{260}$$

with

$$\hat{\mathfrak{A}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{261}$$

The transposed reciprocal matrix $\hat{\mathfrak{z}}(t) = (\hat{\mathfrak{y}}^{-1}(t))^T$ will then be

$$\hat{\mathfrak{z}}(t) = \hat{\Psi}(t) \cdot e^{-\hat{\mathfrak{A}}^T t} = \frac{1}{3+\sin^2 t-\cos t} \cdot \begin{bmatrix} 4\sin t & -\cos t \\ -1+3\cos t & \sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}. \tag{262}$$

The elements of the last row

$$\hat{z}_{(1)} = \frac{-t\sin t - 1+3\cos t}{3+\sin^2 t - \cos t} \text{ and } \hat{z}_{[1]} = \frac{\sin t}{3+\sin^2 t-\cos t} \tag{263}$$

form a fundamental system for the homogeneous differential equation adjoint /166

to eq.(259):

$$\overset{*}{L}[\hat{z}] = \hat{z}'' + \left(\frac{2\sin t\cos t + \sin t}{3 + \sin^2 t - \cos t}\,\hat{z}\right)' + \frac{4 + \cos^2 t}{3+\sin^2 t-\cos t}\,\hat{z} = 0. \tag{264}$$

Next, we consider the solutions $\hat{x}(t)$ of the inhomogeneous differential equation

$$L[\hat{x}] = \hat{x}'' - \frac{2\sin t\cos t + \sin t}{3 + \sin^2 t - \cos t}\,\hat{x}' + \frac{4 + \cos^2 t}{3+\sin^2 t-\cos t}\,\hat{x} = f(t) \tag{265}$$

with

$$f(t) = 3 + \sin^2 t - \cos t. \tag{266}$$

Since we have a solution $z_{[1]}$, periodic with P, we can form the integral

$$\int_0^{2\pi} \hat{z}_{[1]}(t)\, f(t)\,dt = \int_0^{2\pi} \sin t\,dt = 0 \tag{267}$$

and find that, for the inhomogeneous differential equation (265), the exceptional case is present. Then, the general solution $\hat{x}(t)$ is obtained in accordance with the variational method for the constants [see eq.(18)], as follows:

$$\begin{aligned}\hat{x}(t) = &- t\sin t + \left(\frac{5}{2} + \frac{1}{2}\cos 2t - \cos t\right) + \\ &+ t\cdot {}^1c_0 \sin t + {}^1c_0(1-3\cos t) + {}^1c_1 \sin t.\end{aligned} \tag{268}$$

with the derivative

$$\begin{aligned}\hat{x}'(t) = &- t\cos t - \sin 2t + t\cdot {}^1c_0 \cos t + \\ &+ {}^1c_0 \cdot 4\sin t + {}^1c_1 \cos t.\end{aligned}$$

If ${}^1c_0 = 1$ is selected, then $\hat{x}(t)$ is periodic with 2π at arbitrary ${}^1c_1$. However, if ${}^1c_0 \neq 1$, the power order of $\hat{x}(t)$ will be $\hat{m} = 1$. Since the factor of t in $\hat{x}(t)$ is a nonconstant function, periodic with 2π, also the derivative $\hat{x}'(t)$ has the same power order $m' = m = 1$ (see theorem 3, resp. theorem 20).

We then turn to the differential equation /167

$$L[x] = x^{(3)} - \frac{2\sin t\cos t + \sin t}{3 + \sin^2 t - \cos t}\,x^{(2)} + \frac{4 + \cos^2 t}{3 + \sin^2 t - \cos t}\,x^{(1)} = 3 + \sin^2 t - \cos t \tag{269}$$

at $j \geq 1$ which is transformed, with $x^{(j)} = \hat{x}$, into the reduced differential equation (265). First, we construct a fundamental solution matrix $\mathfrak{N}(t)$, according to the method described in Section 5, for the differential equation homogeneous to eq.(269). By selecting successively $j = 1, 2, 3$, we obtain the solution matrices [see eqs.(108), (109), (114)]

$$j=1 \quad \mathfrak{N}(t) = \Phi(t) \cdot \mathfrak{E}(t) = \begin{bmatrix} 1 & -\cos t, -2\sin t \\ 0 & \hat{\Phi} \end{bmatrix} \begin{bmatrix} 1 & 0 & t \\ & 1 & t \\ 0 & & 1 \end{bmatrix}, \tag{270}$$

$$\text{with } \mathfrak{E}(t) = e^{\mathfrak{K}t} \quad \text{and} \quad \mathfrak{K} = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ 0 & & 0 \end{bmatrix},$$

$$j=2 \quad \mathfrak{N}(t) = \Phi(t) \cdot \mathfrak{E}(t) = \begin{bmatrix} 1 & 0 & -\sin t, \cos t \\ & 1 & -\cos t, -2\sin t \\ 0 & & \hat{\Phi} \end{bmatrix} \begin{bmatrix} 1 & t & 0 & \frac{t^2}{2} \\ & 1 & 0 & t \\ & & 1 & t \\ 0 & & & 1 \end{bmatrix}, \tag{271}$$

$$\text{with } \mathfrak{E}(t) = e^{\mathfrak{K}t} \quad \text{and} \quad \mathfrak{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ 0 & & & 0 \end{bmatrix},$$

$$j=3 \quad \mathfrak{N}(t) = \Phi(t) \cdot \mathfrak{E}(t) = \begin{bmatrix} 1 & 0 & 0 & \cos t, 0 \\ & 1 & 0 & -\sin t, \cos t \\ & & 0 & -\cos t, -2\sin t \\ 0 & & & \hat{\Phi} \end{bmatrix} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & 0 & \frac{t^3}{3!} \\ & 1 & t & 0 & \frac{t^2}{2!} \\ & & 1 & 0 & t \\ & & & 1 & t \\ 0 & & & & 1 \end{bmatrix}, \tag{272}$$

$$\text{with } \mathfrak{E}(t) = e^{\mathfrak{K}t} \quad \text{and} \quad \mathfrak{K} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}.$$

Here, according to eq.(260), we have /168

$$\hat{\Phi} = \begin{bmatrix} \sin t & 1-3\cos t \\ \cos t & 4\sin t \end{bmatrix} \tag{273}$$

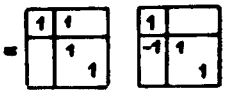
Next, we determine the matrix $\mathfrak{S}$ for the transformation [see eq.(123)]

$$\mathfrak{K}^0 = \mathfrak{L}^{-1} \mathfrak{K} \mathfrak{L},$$

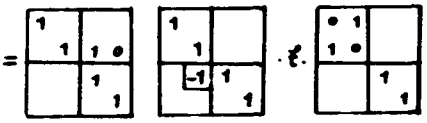
which brings the matrix $\mathfrak{K}$ to the Jordan normal form $\mathfrak{K}^0$. This then yields [see

eqs.(148), (137), (146), (139)]

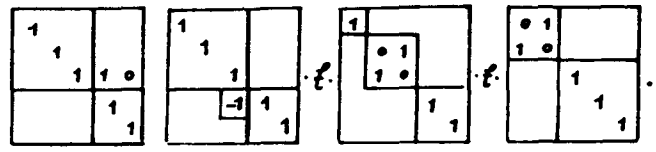
$$j=1 \quad \mathcal{L} = \mathcal{L}_1 = \mathcal{L}_1^* \mathcal{E}_1 = \tag{274}$$



$$j=2 \quad \mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 = \mathcal{L}_1^* \mathcal{E}_1 \cdot \mathcal{L}_2^* \mathcal{U}_2 = \tag{275}$$



$$j=3 \quad \mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3 = \mathcal{L}_1^* \mathcal{E}_1 \cdot \mathcal{L}_2^* \mathcal{U}_2 \cdot \mathcal{L}_3^* \mathcal{U}_3 = \tag{276}$$



We use this for calculating the new fundamental systems [see eqs.(124) and (192)] /169

$$\mathcal{Y}^0(t) = \Phi^0(t) \cdot e^{\mathcal{R}^0 t}$$

$$j=1 \quad \mathcal{Y}^0(t) = \begin{pmatrix} \cos t & 1-\cos t & -2\sin t \\ -\sin t & \sin t & 1-3\cos t \\ -\cos t & \cos t & 4\sin t \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ & 1 \;\; t \\ & 1 \end{pmatrix}, \tag{277}$$

$$j=2 \quad \mathcal{Y}^0(t) = \begin{pmatrix} \sin t & 1 & -3\sin t & \cos t \\ \cos t & 0 & 1-\cos t & -2\sin t \\ -\sin t & 0 & \sin t & 1-3\cos t \\ -\cos t & 0 & \cos t & 4\sin t \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 \;\; t & \frac{t^2}{2!} \\ & 1 \;\; t & \\ & 1 & \end{pmatrix}, \tag{278}$$

$$j=3 \quad \mathcal{Y}^0(t) = \begin{pmatrix} -\cos t & 1 & 0 & \cos t & 0 \\ \sin t & 0 & 1 & -\sin t & \cos t \\ \cos t & 0 & 0 & 1-\cos t & -2\sin t \\ -\sin t & 0 & 0 & \sin t & 1-3\cos t \\ -\cos t & 0 & 0 & \cos t & 4\sin t \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 \;\; t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ & 1 \;\; t & \frac{t^2}{2!} & \\ & 1 \;\; t & & \\ & 1 & & \end{pmatrix}. \tag{279}$$

In eq.(277), it must also be noted that

$$j = 1,\ i_1 = 1,\ \text{i.e.,}\ j = i_1,\ \gamma = 1\ \text{[see eq.(195)]}$$

We read from this:

$$\varphi_{\alpha 0} = -\tilde{\varphi}_{(1)} = \cos t \neq \text{Const.},\quad \varphi_{\alpha(1)} = 1 + \tilde{\varphi}_{(1)} = 1 + \cos t \neq \text{Const.},\quad \varphi_{\alpha(1)+1} = \tilde{\varphi}_{(1)+1} = -2\sin t.$$

This result coincides with theorems 19 and 23. Analogously, the following is valid with respect to eq.(278):

$$j = 2,\ j > i_1,\ \gamma = 1$$

Equation (278) indicates that

$$\varphi_{\alpha 0} = -\tilde{\varphi}_{(1)} = \sin t + \text{Const.},$$

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$$\varphi_{\alpha(1)} = \varphi_{\alpha(1)} = 1,\ \varphi_{\alpha(1)+j-i_2} = \varphi_{\alpha(1)+1} = \tilde{\varphi}_{(1)} = -\sin t \neq \text{Const.},\ \varphi_{\alpha(1)+2} = \tilde{\varphi}_{(1)+1} = \cos t.$$

This result also coincides with theorems 19 and 23. With respect to eq.(279), we have

$$j = 3,\ j > i_1 = 1,\ \gamma = 1,$$

$$\varphi_{\alpha(1)} = -\tilde{\varphi}_{(1)} = -\cos t \neq \text{Const.},$$

$$\varphi_{\alpha(1)} = \varphi_{\alpha(1)} = 1,\ \varphi_{\alpha(1)+1} = 0,\ \varphi_{\alpha(1)+j-i_2} = \varphi_{\alpha(1)+2} = \tilde{\varphi}_{(1)} = \cos t \neq \text{Const.},\ \varphi_{\alpha(1)+3} = \tilde{\varphi}_{(1)+1} = 0,$$

which again coincides with theorems 19 and 23.

The reciprocal transposed matrices, conjugate with eqs.(277) - (279),

$$\mathfrak{Z}^{\circ} = \left(\mathfrak{W}^{\circ -1}\right)^{T} = \left(\mathfrak{F}^{-1}\right)^{T} \cdot e^{-\mathfrak{K}^{\circ T} t} = \mathfrak{Z}^{\circ} \cdot e^{-\mathfrak{K}^{T} t}$$

will then read

$$j=1 \left\| \mathfrak{Z}^{\circ}(t) = \frac{1}{3 + \sin^2 t - \cos t} \begin{vmatrix} 3 + \sin^2 t - \cos t & 3 + \sin^2 t - \cos t & 0 \\ 2\sin t \cos t - 4\sin t & 2\sin t \cos t & -\cos t \\ 3 + \cos^2 t - 4\cos t & 2 + \cos^2 t - \cos t & \sin t \end{vmatrix} \begin{vmatrix} 1 & \\ & 1 \\ & -t\ \ 1 \end{vmatrix}, \right. \tag{280}$$

$$j=2 \left\| \mathfrak{Z}^{\circ}(t) = \frac{1}{3 + \sin^2 t - \cos t} \right. \begin{vmatrix} 0 & 3 + \sin^2 t - \cos t & 0 & 0 \\ 3 + \sin^2 t - \cos t & 0 & 3 + \sin^2 t - \cos t & 0 \\ 2\sin t \cos t - 4\sin t & 1 + 3\sin^2 t & 2\sin t \cos t & -\cos t \\ 3 + \cos^2 t - 4\cos t & -\sin t + 2\sin t \cos t & 2 + \cos^2 t - \cos t & \sin t \end{vmatrix} \times \begin{vmatrix} 1 & & \\ & 1 & \\ & -t & 1 \\ & \frac{t^2}{2!} & -t\ \ 1 \end{vmatrix}. \tag{281}$$

$j=3 \left\| \mathfrak{F}^{\circ}(\omega) = \frac{1}{3+\sin^2 t - \cos t} \right.$. /171

$\circ$	$3+\sin^2 t-\cos t$	$\circ$	$\circ$	$\circ$
$\circ$	$\circ$	$3+\sin^2 t-\cos t$	$\circ$	$\circ$
$3+\sin^2 t-\cos t$	$\circ$	$\circ$	$3+\sin^2 t-\cos t$	$\circ$
$2\sin t\cos t-4\sin t$	$-4\sin t\cos t$	$1+3\sin^2 t$	$2\sin t\cos t$	$-\cos t$
$3+\cos^2 t-4\cos t$	$\cos t-3\cos^2 t$	$-\sin t+2\sin t\cos t$	$2+\cos^2 t-\cos t$	$\sin t$

1				
	1			
	$-t$	1		
	$\frac{t^2}{2!}$	$-t$	1	
	$-\frac{t^3}{3!}$	$\frac{t^2}{2!}$	$-t$	1

From this, one can read from the last row

$$z_{[[0]]} = \frac{1}{3+\sin^2 t-\cos t}\left(\frac{7}{2} - 4\cos t + \frac{1}{2}\cos 2t\right) \text{ for } j=1,2,3, \tag{283}$$

i.e., independent of j in accordance with the proof of theorem 14. The corresponding periodic solution $z_{[[1]]}$, in all three cases, will be (see theorem 13)

$$z_{[[1]]} = \hat{z}_{[1]} = \frac{\sin t}{3 + \sin^2 t - \cos t} . \tag{284}$$

Naturally, the function $z_{[[0]]}$ can also be calculated in accordance with the formula (163):

$$z_{[[0]]} = -\left(\hat{\Psi}_{(1)}\, {}_j\tilde{\Psi}_{(1)} + \hat{\Psi}_{[1]}\, {}_j\tilde{\Psi}_{[1]} + \hat{\Psi}_{(1)}\right) . \tag{285}$$

In accordance with the last row of the matrix $\hat{\Psi}$ [see eq.(262)], we have the functions

$$\hat{\Psi}_{(1)} = \frac{-1 + 3\cos t}{3 + \sin^2 t - \cos t} \text{ and } \hat{\Psi}_{[1]} = \frac{\sin t}{3+\sin^2 t-\cos t}$$

and, in accordance with the j^{th} power row of the matrix Φ [see eqs.(270)-(272)], the functions

$${}_j\tilde{\Psi}_{(1)} = -\cos t \text{ and } {}_j\tilde{\Psi}_{[1]} = -2\cdot\sin t .$$

As readily verified, this will also yield eq.(283).

In addition, it is possible to read from the representation of eqs.(277) /172 to (279) resp. eqs.(280) - (282), the orders m_ν of the elementary components, resulting in the scheme

	$j = 0$	1	2	3	$j \geqslant 2$
m_0	0	1	1	1	1
m_1	2	2	3	4	$j+1$

(286)

which coincides accurately with the scheme of the minimal orders for the resonance subcase [see eq.(185)].

For our special case (266), the quantity $\nu = 1$ in accordance with eq.(267) is the exceptional index. Conversely, a computation of the integral [see eq.(283)]

$$\int_0^{2\pi} z_{[\nu=j]}(t) f(t)\,dt = \pi\pi \neq 0, \tag{287}$$

shows that $\nu = 0$ is a resonance index.

Naturally, instead of eq.(287) one can also calculate (see theorem 18) the integral

$$\int_0^{2\pi} z_{[\nu=j]}(t) f(t)\,dt = \int_0^{2\pi} \hat{x}(t)\,dt = \pi\pi \neq 0$$

by means of the solution

$$\hat{x}(t) = \frac{7}{2} - 4\cos t + \frac{1}{2}\cos 2t + {}^{1}c_1 \sin t \quad \text{see eq.(268) with } {}^{1}c_0 = 1) \tag{288}$$

of eq.(265), periodic with 2π. In this manner, the following sequence of values is obtained for the overall minimal power orders of the solutions $x(t)$ for $j = 0, 1, 2, 3$, etc.:

$$m = m_0 = 0,1,1,1, \text{ etc.} \tag{289}$$

The general solution $x(t)$ of eq.(269) can be determined in two different ways: either by a successive integration, starting from $\hat{x}(t)$, to eq.(288) or by /173 constructing a particular solution in accordance with the method of variation of the constants on eq.(269) and adding the general solution of the conjugate homogeneous differential equation which must be taken from the matrix $\mathfrak{M}^{\rho}(t)$ [see eqs.(277) - (279)].

By successive integration of eq.(268) (j = 1, 2, 3), we obtain the solutions

$$\begin{cases} j=1 \;\|\; x(t) = t\left(\frac{5}{2}+\cos t\right) - \left(2\sin t - \frac{1}{4}\sin 2t\right) + \\ t\,{}^1c_0\,(1-\cos t) + \left(-2\,{}^1c_0\sin t - {}^1c_1\cos t + a_1\right), \\ j=2 \;\|\; x(t) = \frac{5}{2}\cdot\frac{t^2}{2!} + t\sin t + \left(3\cos t - \frac{1}{8}\cos 2t\right) + \\ \frac{t^2}{2!}\,{}^1c_0 + t\left(a_1 - {}^1c_0\sin t\right) + \left({}^1c_0\cos t - {}^1c_1\sin t + a_2\right), \\ j=3 \;\|\; x(t) = \frac{5}{2}\cdot\frac{t^3}{3!} - t\cos t + \left(4\sin t - \frac{1}{16}\sin 2t\right) + \\ \frac{t^3}{3!}\,{}^1c_0 + \frac{t^2}{2!}\,a_1 + t\left({}^1c_0\cos t + a_2\right) + \left({}^1c_1\cos t + a_3\right). \end{cases} \tag{290}$$

The variational method for the constants, using the argument

$$x(t) = {}^0x(t) + {}^1x(t) \tag{291}$$

and [see eq.(20)]

$$x(t) = \sum_{\mu=0}^{[\nu]} \varphi_\mu(t)\cdot\int^t z_\mu(t)\,f(t)\,dt \quad \text{for } \nu = 0,1 \tag{292}$$

would yield

$$\dot{x}=1 \;\|\; \begin{cases} {}^0x(t) = \cos t\left(\frac{7}{2}t - 4\sin t + \frac{1}{4}\sin 2t + {}^0c_1\right), \\ {}^1x(t) = \frac{5}{2}t(1-\cos t) + \left(\frac{9}{2}\sin t\cos t - 2\sin t - \frac{1}{4}\sin 2t\cos t\right)+ \\ \quad + t\,{}^1c_0\,(1-\cos t) - 2\,{}^1c_0\sin t + {}^1c_1\,(1-\cos t), \\ x(t) = t\left(\frac{5}{2}+\cos t\right) - \left(2\sin t - \frac{1}{4}\sin 2t\right) + \\ \quad + t\cdot{}^1c_0(1-\cos t) + \left(-2\,{}^1c_0\sin t - ({}^1c_1 - {}^0c_1)\cos t + {}^1c_1\right), \end{cases} \tag{293}$$

which coincides with the first equation of the system (290) when taking into /174 consideration that ${}^1c_0$, ${}^1c_1$, and ${}^0c_1$, or a_1 are arbitrary constants. Analogously, we obtain for

$$\dot{x}=2 \;\|\; \begin{cases} {}^0x(t) = \sin t\left(\frac{7}{2}t - 4\sin t + \frac{1}{4}\sin 2t + {}^0c_1\right), \\ {}^1x(t) = \frac{5}{2}\left(\frac{t^2}{2!} - t\sin t\right) + \left(3\cos t - \frac{1}{8}\cos 2t + \right. \\ \quad \left. 4\sin^2 t - \frac{1}{4}\sin t\cdot\sin 2t\right) + \frac{t^2}{2}\cdot{}^1c_0 + \\ \quad t\left({}^1c_1 - {}^1c_0\sin t\right) + \left({}^1c_0\cos t - \right. \end{cases} \tag{294}$$

$$
\left\|\begin{array}{l}
\qquad {}^1c_1 \sin t + {}^1c_2),\\
x(t) = \frac{5}{2}\cdot\frac{t^2}{2!} + t\sin t + (3\cos t - \frac{1}{8}\cos 2t)\\
\qquad + \frac{t^2}{2}\cdot {}^1c_0 + t\,({}^1c_1 - {}^1c_0\sin t) +\\
\qquad {}^1c_0\cos t - ({}^1c_1 - {}^0c_1)\sin t + {}^1c_2
\end{array}\right.
$$

and for

$$
j=3\left\|\begin{array}{l}
{}^0x(t) = -\cos t\,(\frac{7}{2}t - 4\sin t + \frac{1}{4}\sin 2t + {}^0c_1),\\
{}^1x(t) = \frac{5}{2}(\frac{t^3}{3!} + t\cos t) + 4\sin t -\\
\qquad \frac{33}{16}\sin 2t + \frac{1}{4}\sin 2t\cos t +\\
\qquad {}^1c_0(\frac{t^3}{3!} + t\cos t) + {}^1c_1(\frac{t^2}{2!} + \cos t) +\\
\qquad {}^1c_2 t + {}^1c_3\,,\\
x(t) = \frac{5}{2}\cdot\frac{t^3}{3!} - t\cos t + (4\sin t - \frac{1}{16}\sin 2t) +\\
\qquad \frac{t^3}{3!}{}^1c_0 + \frac{t^2}{2!}{}^1c_1 + t({}^1c_0\cos t + {}^1c_2) +\\
\qquad ({}^1c_1 - {}^0c_1)\cos t + {}^1c_3.
\end{array}\right. \qquad (295)
$$

Again, we obtain coincidence with the two last relations of the system (290). /175

Solutions with the indicated minimal order (289) are readily constructed [see eqs.(293) - (295)]:

For $j = 0$, we only must put (as already known) ${}^1c_0 = 1$ in eq.(268).

For $j = 1$, the selection of constants in eq.(290) is arbitrary.

For $j = 2$, we put ${}^1c_0 = \frac{5}{2}$ while the other constants are arbitrary.

For $j = 3$, we put ${}^1c_0 = -\frac{5}{2}$ and ${}^1c_1 = 0$, while the other constants are arbitrary.

If we start from a fixed solution $x(t)$, power orders are obtained for the sequence of the functions $x(t)$, $x'(t)$, $x''(t)$, $x'''(t)$, etc., whose patterns can be determined in accordance with eq.(290). Let us consider the case $j = 3$ and

select a definitely determined $x(t)$, resulting in the power orders of the derivatives $x^{(k)}(t)$ for four different cases as shown in Figs.14 - 17 [see eq.(295)].

In all these cases, the following is valid:

$$\left.\begin{aligned} &j = 3,\ i_1 = 1,\ \gamma = 1 \text{ [see eq.(195)]} \\ &\omega_0 = m_0 = 1 \quad [\nu = 0 \text{ resonance index, see eq.(287)}] \end{aligned}\right\} \tag{296}$$

Case I: ${}^1c_0 + \frac{5}{2} \neq 0$.

Here, we have $\omega_1 = 3$, $\ell_\gamma = j - i_\gamma = 2$ [see eqs.(204) and (296)]. The pertaining pattern corresponds to Fig.7.

/176

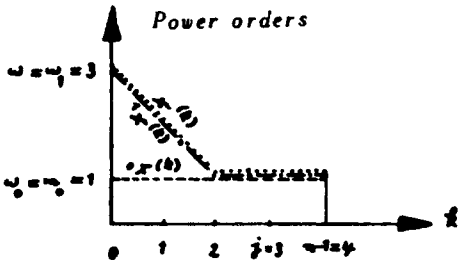


Fig.14

Case II: ${}^1c_0 + \frac{5}{2} = 0$, but ${}^1c_1 \neq 0$.

Here, we have $\omega_1 = 2$, $\ell_\gamma = 2$.

The following pattern corresponds to Fig.8:

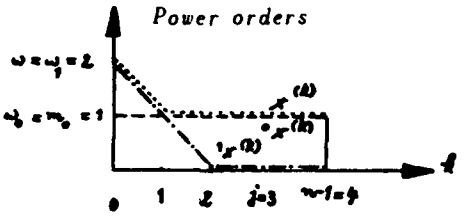


Fig.15

Case III: ${}^1c_0 + \frac{5}{2} = 0$, ${}^1c_1 = 0$, and ${}^1c_2$ arbitrary.

Here, we have $\omega_1 = 1$, $\ell_\gamma = \ell_1 = \text{Min}\,(j - i_1, \omega_1) = \text{Min.}\,(2, 1) = 1$ [see eqs.(204) and (296)].

In accordance with Fig.6, we obtain here

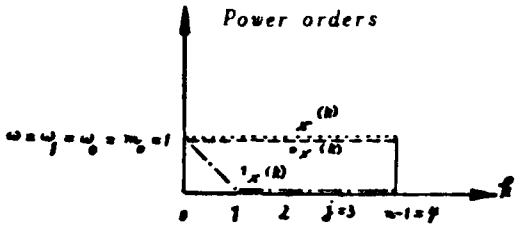


Fig.16

Case IV: ${}^1c_0 - 1 = 0$, and the remaining c arbitrary. /177

Here, we have

$$\begin{cases} \omega_\gamma = \omega_1 = 3,\ l_\gamma = j - i_\gamma = 2, \\ \overline{\gamma} = 0 \text{ (see eq.(220))},\ i_{\overline{\gamma}} = i_0 = 0 \text{ (see eq.(152))}, \\ \omega_{\overline{\gamma}} = m_{\overline{\gamma}} = \omega_0 = m_0 = 1 \text{ (see eq.(296))}, \\ l = j - i_{\overline{\gamma}} = 3 - 0 = 3 \text{ (see eq.(226))}. \end{cases} \tag{297}$$

The constant ${}^1c_0$ is so selected that [see eq.(291)] the coefficient of t vanishes in the sum $x'''(t) = {}^0x'''(t) + {}^1x'''(t)$ (see theorem 24).

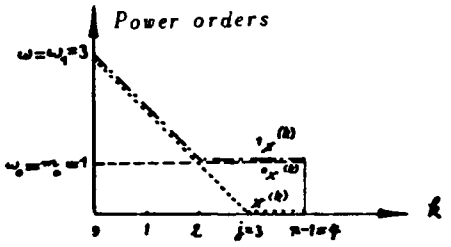


Fig.17

See also Fig.10.

In this case, the condition (242) must be satisfied. We will check this condition; the only remaining item is

$${}^1d_0 = {}^0d_0 . \tag{298}$$

where ${}^1d_0$ is the coefficient of the highest power in ${}^1x$ [see eqs.(122) resp. (199) and (26)], i.e., $\frac{5}{2} + {}^1c_0$ according to eq.(295), whereas ${}^0d_0$ is the constant factor of the highest power in ${}^0x(t)$, i.e., equal to $\frac{7}{2}$ according to eq.(295). In accordance with eq.(203) resp. (26), the coefficient of ${}^0x(t)$ can be computed [see eq.(279)]:

$$ {}^{\nu}\theta_0(t) = {}^0\theta_0(t) = {}^0d_0\ \varphi_{(\cos)}(t) = {}^0d_0\ (-\cos t), $$

so that, necessarily, ${}^0d_0 = \frac{7}{2}$. Consequently, the condition (298) reads as follows:

$$ \frac{5}{2} + {}^1c_0 = \frac{7}{2}, \text{ which yields } {}^1c_0 = 1 \qquad \text{/178} $$

as had been assumed above, for the case IV.

The four traces from Figs.14 - 17 are compiled in Fig.18, together with the minimal solution of Fig.18 (the latter is shown as a solid line).

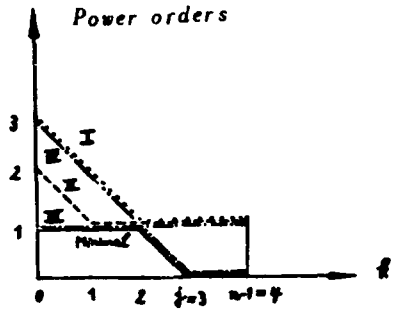


Fig.18

It is readily demonstrated that all possible cases are covered by the cases I - IV. No solution x(t) exists which, simultaneously with all three derivatives $x'(t)$, $x''(t)$, and $x'''(t)$, would have the corresponding minimal order, as represented in Fig.18 by the solid line.

As a second example, we will consider the differential equation

$$ x^{(j+3)} + \frac{2\cos t}{3-2\sin t}\, x^{(j+2)} + x^{(j+1)} + \frac{2\cos t}{3-2\sin t}\, x^{(j)} = f(t) \quad (j \geq 0) \qquad (299) $$

with two different right-hand sides: /179

$$\text{(a)}\quad f(t) = \sin t\,(2\sin t - 3),$$
$$\text{(b)}\quad f(t) = (\sin t - 1)\,(2\sin t - 3).$$

The general solution of the inhomogeneous reduced differential equation ($j = 0$) has the following form:

$$\hat{x}(t) = \hat{x}^{*}(t) + {}^{1}c_1\,\hat{y}_{(1)}(t) + {}^{1}c_0\,\hat{y}_{[1]}(t) + {}^{2}c_1\,\hat{y}_{(2)}(t) \tag{300}$$

with

$$\begin{cases} \hat{y}_{(1)}(t) = \hat{\varphi}_{(1)}(t) = \frac{1}{3}\cos t, \\ \hat{y}_{[1]}(t) = t\cdot\hat{\varphi}_{(1)}(t) + \hat{\varphi}_{[1]}(t) = \\ \qquad = t\cdot\frac{1}{3}\cos t + (\frac{1}{3}\sin t + 1), \\ \hat{y}_{(2)}(t) = \hat{\varphi}_{(2)}(t) = \sin t. \end{cases} \tag{301}$$

According to theorems 25 - 27, as already expressed in the notation,

$$\begin{cases} \hat{m}_1 = 2,\ \hat{m}_2 = 1,\ i_1 = 1,\ i_2 \text{ is nonexistent}, \\ \alpha_1 = \alpha_2 = 0, \\ \hat{\varphi}_{(1)} = \frac{1}{3}\cos t,\ \hat{\varphi}_{[1]} = \frac{1}{3}\sin t + 1,\ \hat{\varphi}_{(2)} = \sin t. \end{cases} \tag{302}$$

Further, for the case (a) we calculate

$$\hat{x}^{*}(t) = t(\tfrac{3}{2}\sin t - \cos t) + (\tfrac{1}{3}\sin t + 3\cos t + \tfrac{2}{3}\sin t\cos t - 3) \tag{303}$$

and, for the case (b), /180

$$\hat{x}^{*}(t) = \frac{t^2}{2}\cos t + t\,(\sin t - \cos t + 3) + (\tfrac{11}{4}\cos t - \tfrac{8}{3}\sin t + \tfrac{2}{3}\sin t\cos t - 3). \tag{304}$$

By means of the matrix $\hat{\mathfrak{Z}}(t) = \left(\hat{\mathfrak{Y}}^{-1}(t)\right)^{T}$, the following is obtained:

$$\begin{cases} \hat{z}_{(1)} = -t\,\hat{\gamma}_{[1]} + \hat{\gamma}_{(1)} = -t\cdot\frac{-3}{2\sin t - 3} + \\ \qquad + \frac{3(3\cos t - \sin t\cos t)}{2\sin t - 3}, \\ \hat{z}_{[1]} = \hat{\gamma}_{[1]} = \frac{-3}{2\sin t - 3}, \end{cases} \tag{305}$$

$$\left| \hat{z}_{[2]} \cdot \hat{\varphi}_{[2]} = \frac{1 + 3\sin t + \cos^2 t}{2\sin t - 3} \right.$$

For the case (a), this will yield

$$\left\{ \begin{array}{l} \int_0^{2\pi} \hat{z}_{[1]}(t)\, f(t)\,dt = -\int_0^{2\pi} 3\sin t\,dt = 0, \\ \qquad\qquad \nu = 1 \text{ exceptional index} \\ \text{and} \\ \int_0^{2\pi} \hat{z}_{[2]}(t) f(t)\,dt = \int_0^{2\pi} \sin t\,(1+3\sin t + \cos^2 t)\,dt = \\ \qquad = 3\pi \neq 0,\ \nu = 2 \text{ resonance index} \end{array} \right. \tag{306}$$

and, for the case (b),

$$\left\{ \begin{array}{l} \int_0^{2\pi} \hat{z}_{[1]}(t) f(t)\,dt = -3\int_0^{2\pi} (\sin t - 1)\,dt = \\ \qquad = 6\pi \neq 0,\ \nu = 1 \text{ resonance index} \\ \text{and} \\ \int_0^{2\pi} \hat{z}_{[2]}(t) f(t)\,dt = \int_0^{2\pi} (\sin t - 1)(1+3\sin t + \cos^2 t)\,dt = \\ \qquad = 0,\ \nu = 2 \text{ exceptional index} \end{array} \right. \tag{307}$$

The solution $z_{[[0]]}(t)$ is calculated according to the formula (163):

$$z_{[[0]]}(t) = -\left(\hat{\gamma}_{(1)} \,{}_1\tilde{\varphi}_{(1)} + \hat{\gamma}_{[1]} \,{}_1\tilde{\varphi}_{[1]} + \hat{\gamma}_{(2)} \,{}_1\tilde{\varphi}_{(2)} + \hat{\gamma}_{(1)} \right). \tag{308}$$

The still to be determined functions ${}_1\tilde{\varphi}_{(1)}$, ${}_1\tilde{\varphi}_{[1]}$, ${}_1\tilde{\varphi}_{(2)}$ are defined as functions with zero mean value from the system of differential equations (97) which, in our case, has the following appearance: /181

$$\left\{ \begin{array}{lll} \tilde{\varphi}_{(1)}' = \tilde{\tilde{\varphi}}_{(1)} & , & \tilde{\varphi}_{(1)}' = \frac{1}{3}\cos t \\ \tilde{\varphi}_{[1]}' = \tilde{\hat{\varphi}}_{[1]} - \tilde{\varphi}_{(1)} & , & \tilde{\varphi}_{[1]}' = \frac{1}{3}\sin t - \tilde{\varphi}_{(1)} \\ \tilde{\varphi}_{(2)}' = \tilde{\hat{\varphi}}_{(2)} & , & \tilde{\varphi}_{(2)}' = \sin t\,. \end{array} \right. \tag{309}$$

As solutions, we obtain

$$\tilde{\varphi}_{(1)} = \frac{1}{3}\sin t,\ \tilde{\varphi}_{[1]} = 0,\ \tilde{\varphi}_{(2)} = -\cos t.$$

On substituting this, together with eq.(305), into eq.(308), we obtain

$$z_{[0]} = -\frac{2\cos t - 3\sin t\cos t}{2\sin t - 3}. \tag{310}$$

In addition, according to eqs.(162) and (305), we have

$$z_{[1]} = \hat{z}_{[1]} = \hat{Y}_{[1]} = \frac{-3}{2\sin t - 3} \tag{311}$$

as well as

$$z_{[2]} = \hat{z}_{[2]} = \hat{Y}_{[2]} = \frac{1 + 3\sin t + \cos^2 t}{2\sin t - 3} \tag{312}$$

In both cases (a) and (b), the exceptional case exists for $\nu = 0$.

Thus, the scheme of the minimal orders, according to eq.(185), will be

	$j = 0$	1	2	3	4	$j \geq 2$
m_0	0	1	1	1	1	1
m_1	2	2	3	4	5	$j + 1$
m_2	1	1	1	1	1	1

From this, in the case (a), we obtain for $j = 0, 1, 2, 3, \ldots$ the sequence /182
of minimal orders [see eq.(184)]:

$$m = m_2 = 1,1,1,1,\ldots . \tag{313}$$

In the case (b), we obtain accordingly

$$m = m_1 = 2,2,3,4,\ldots . \tag{314}$$

Then, we determine the general solution $x(t)$ for $j = 4$, yielding:

In the case (a):

$$x(t) = -\frac{3t^4}{41} + t\left(\frac{3}{2}\sin t - \cos t\right) + \left(\frac{13}{3}\sin t + \right.$$
$$\left. + 9\cdot\cos t + \frac{1}{48}\sin 2t\right) + {}^1c_0\frac{t^4}{41} + a_1\frac{t^3}{31} + a_2\frac{t^2}{21} +$$
$$+ t\left({}^1c_0\cdot\frac{1}{3}\cos t + a_3\right) + \left(-{}^1c_0\sin t + {}^1c_1\cdot\frac{1}{3}\cos t + \right.$$
$$\left. + t^2{}^1c_1\sin t + a_4\right).$$

In the case (b):

$$x(t) = 3\frac{t^5}{5!} - 3\frac{t^4}{4!} + \frac{t^2}{2!}\cos t + t(-3\sin t - \cos t) +$$
$$+\left(\frac{4}{3}\sin t - \frac{13}{4}\cos t + \frac{16}{3}\sin 2t\right) + {}^1c_1\frac{t^4}{4!} + a_1\frac{t^3}{3!} +$$
$$+a_2\frac{t^2}{2!} + t\left({}^1c_1\cdot\frac{1}{3}\cos t + a_3\right) + \left(-{}^1c_1\sin t +\right.$$
$$\left.+{}^1c_2\cdot\frac{1}{3}\cos t + {}^2c_0\sin t + a_4\right).$$

In the case (a), four subcases must be differentiated:

$${}^1c_0 - 3 \neq 0,$$
$${}^1c_0 - 3 = 0, \text{ but } a_1 \neq 0,$$
$${}^1c_0 - 3 = a_1 = 0, \text{ but } a_2 \neq 0,$$
$${}^1c_0 - 3 = a_1 = a_2 = 0,$$

from which the following curves for the power orders, analogous to Fig.18, are obtained:

/183

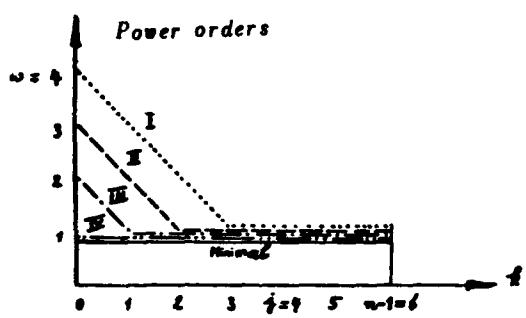
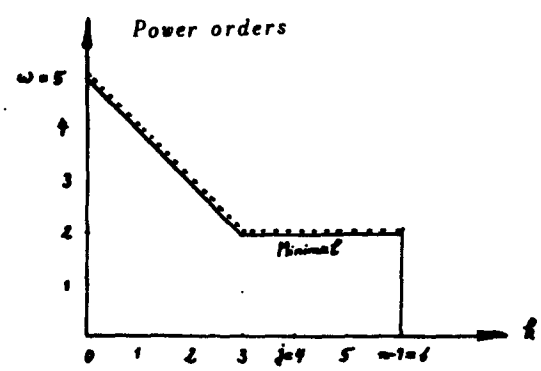


Fig.19

In the case (b) only one possible case exists with arbitrary constants, yielding the pattern shown in Fig.20:



It should be mentioned, in addition, that it would not have been necessary in the above examples to check the index $\nu = 0$ as to resonance since the minimal order m_0 can never be greater than m_1 or m_2.

Example 3: As the third example, let us consider the differential equation

$$x^{(j+3)} - \frac{4 + \sin 2t + 2\cos 2t}{\sin 2t + 4} x^{(j+2)} + \\ + \frac{6 + 2\cos 2t - \sin 2t}{\sin 2t + 4} x^{(j+1)} + \frac{\sin 2t - 6}{\sin 2t + 4} x^{(j)} = f(t) \quad (j \geq 0) \tag{315}$$

with the right-hand sides /184

$$\text{(c)} \quad f(t) = \cos t\,(4 + 2\sin t\cos t),$$
$$\text{(d)} \quad f(t) = (4 + 2\sin t\cos t).$$

The general solution of the inhomogeneous reduced differential equation has the following form:

$$\hat{x}(t) = \hat{x}^*(t) + {}^1c_2\,\hat{y}_{(1)}(t) + {}^1c_1\,\hat{y}_{[1]}(t) + {}^2c_0\,\hat{y}_{(2)}(t) \tag{316}$$

with

$$\begin{cases} \hat{y}_{(1)}(t) = \hat{\varphi}_{(1)}(t) = \cos t \\ \hat{y}_{[1]}(t) = t\,\hat{\varphi}_{(1)}(t) + \hat{\varphi}_{[1]}(t) = t\cos t + \sin t \\ \hat{y}_2(t) = \hat{\varphi}_2(t) \cdot e^t = e^t, \end{cases} \tag{317}$$

where, in the case (c),

$$\hat{x}^*(t) = \frac{9}{10} e^t - \frac{1}{4} t^2 \cos t - t\left(\frac{5}{4}\cos t + \frac{1}{2}\sin t\right) + \\ + \left(\frac{11}{10}\sin t - \frac{11}{20}\cos t - \frac{4}{5}\sin^3 t - \right. \\ \left. - \frac{7}{20}\cos^3 t - \frac{3}{4}\sin t\cos^2 t - \frac{1}{2}\sin^2 t\cos t\right) \tag{318}$$

and, in the case (d),

$$\hat{x}^*(t) = \frac{8}{5} e^t - t\cos t + \left(-\sin t + \frac{1}{5}\sin 2t - \right. \\ \left. - \frac{1}{10}\cos 2t - \sin^2 t - \frac{3}{2}\right) \tag{319}$$

Thus, we have $\hat{m}_1 = 2$, $\hat{m}_2 = 1$ in which case, for $\nu = 2$, the principal case is present; for $\nu = 1$, no i_1 exists. For the matrix $\hat{\mathfrak{Z}}(t) = \left(\hat{\mathfrak{N}}^{-1}(t)\right)^T$, we obtain

$$\begin{cases} \hat{z}_{(1)} = -t\,\hat{\gamma}_{[1]} + \hat{\gamma}_{(1)} = t \cdot \dfrac{\sin t + \cos t}{4 + 2\sin t \cos t} + \\ \quad + \dfrac{\sin t - 2\cos t}{4 + 2\sin t \cos t}, \\ \hat{z}_{[1]} = \hat{\gamma}_{[1]} = -\dfrac{\sin t + \cos t}{4 + 2\sin t \cos t}, \\ \hat{z}_2 = \hat{\gamma}_2\, e^{-t} = \dfrac{1 + \cos^2 t}{4 + 2\sin t \cos t} \cdot e^{-t}. \end{cases} \tag{320}$$

In the case (c), we calculate /185

$$\int_0^{2\pi} \hat{z}_{[1]}(t)\, f(t)dt = -\int_0^{2\pi} (\sin t + \cos t)\cos t\, dt = \\ = -\pi \neq 0, \quad \nu = 1 \text{ resonance index} \tag{321}$$

and, in the case (d),

$$\int_0^{2\pi} \hat{z}_{[1]}(t)\, f(t)dt = -\int_0^{2\pi} (\sin t + \cos t)dt = 0, \\ \nu = 1 \text{ exceptional index} \tag{322}$$

For the orders of the elementary components with variable j, we obtain the scheme

	$j = 0$	1	2	3	$j \geq 0$
m_0	0	1	2	3	j
m_1	2	2	2	2	2
m_2	1	1	1	1	1

In the case (c), we obtain $\nu = 1$ as the resonance index. For establishing that $\nu = 0$ is the exceptional index, it is only necessary to determine $z_{[[0]]}$ which, according to eq.(163), yields

$$z_{[[0]]} = \frac{-2 + 2\sin t \cos t}{4 + 2\sin t \cos t}$$

which means that $\nu = 0$ is the exceptional index. According to eq.(184), the following values are obtained for the minimal orders:

$$m = m_1 = 2,2,2, \quad etc. \tag{323}$$

Thus, in analogy to Fig.18, the pattern shown in Fig.21 will be obtained.

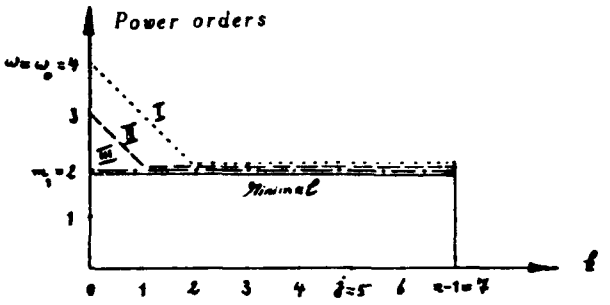


Fig.21

For each curve, a normal solution x(t) can be defined. /186

Case (d): With the period $P = 2\pi$, the quantity $\nu = 0$ is the resonance index, $\nu = 1$ the exceptional index, and $\nu = 2$ the principal index. Since, for the reduced differential equation, the resonance case does not exist, the following can be calculated according to theorem 18:

$$\int_0^{2\pi} z_{[[0]]}(t)\, f(t)dt = \int_0^{2\pi} \hat{x}(t)dt = -4\pi \neq 0$$

with the solution $\hat{x}(t)$, periodic with 2π [see eq.(316)] at ${}^2c_0 = -\frac{\varepsilon}{5}$ and ${}^1c_1 = 1$. For the minimal orders, the following values are obtained:

$$m = m_0 = 0,1,2,3, \text{ etc.}$$

For the power orders with two case differentiations, the pattern of all possible cases is obtained. (In the dotted curve, $\omega_1 = 1$ has been assumed while, in the

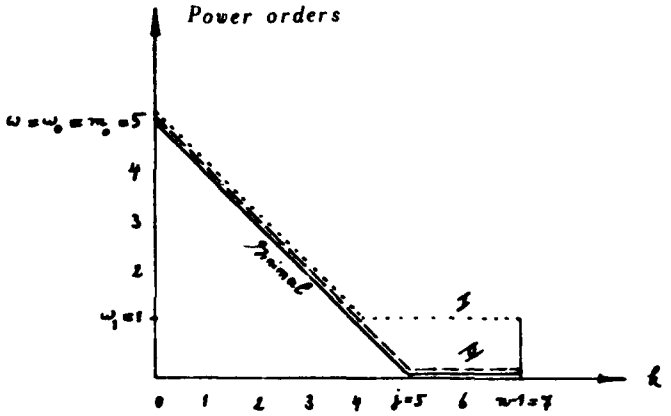


Fig.22

dashed curve, the minimal solution $\omega_1 = 0$ has been assumed.)

Example 4: In the differential equation /187

$$x^{(j+4)} + \frac{\sin t}{2+\cos t} x^{(j+3)} + \frac{4+\cos t}{2+\cos t} x^{(j+2)} + \frac{\sin t}{2+\cos t} x^{j+1} + \frac{2}{2+\cos t} x^{(j)} = -2(2+\cos t) \tag{324}$$

with $j \geq 0$ which, for $j = 0$, has the general solution

$$\hat{x}(t) = \frac{t^2}{2!}\cos t + \frac{1}{2} t \sin t + 2\cos t - 2 + {}^1c_0(t\sin t + 1) + {}^1c_1 \sin t + {}^2c_1 \cdot t\cos t + {}^2c_2 \cdot \cos t \tag{325}$$

we obtain

$$\begin{cases} \hat{\varphi}_{(1)} = \sin t, \ \hat{\varphi}_{[1]} = 1, \ \hat{\varphi}_{(2)} = \cos t, \ \hat{\varphi}_{[2]} = 0, \\ \hat{m}_1 = 2, \ \hat{m}_2 = 2 \text{ mit } i_1 = 1 \text{ and } i_2 \text{ is nonexistent,} \end{cases} \tag{326}$$

Here, our scheme for the minimal orders reads

	$j = 0$	1	2	3	$j \geq 1$
m_0	0	1	1	1	1
m_1	2	2	2	2	2
m_2	2	2	2	2	2

According to eqs.(164) and (166), we calculate

$$\hat{z}_{[1]} = \hat{\gamma}_{[1]} = \frac{-\sin t}{(2+\cos t)} \text{ and } \hat{z}_{[2]} = \hat{\gamma}_{[2]} = -\frac{2\cos t + 1}{2(2+\cos t)} . \tag{327}$$

This will yield

$$\begin{cases} \int_0^{2\pi} \hat{z}_{[1]}(t) f(t) dt = 2 \int_0^{2\pi} \sin t \, dt = 0 , \\ \text{and} \\ \int_0^{2\pi} \hat{z}_{[2]}(t) f(t) dt = \int_0^{2\pi} (2\cos t + 1) dt = 2\pi \neq 0. \end{cases} \tag{328}$$

Consequently, the exceptional case is present for $\nu = 1$, while the resonance case occurs for $\nu = 2$. Thus, the overall minimal order [see eq.(184)] will be $m = \underset{(\nu_{res.})}{\text{Max}}(m_\nu) = 2$ for all j. Here, it is of no importance whether $\nu = 0$ is /188

the resonance index or not (see case IV on p.108). All solutions x(t), no matter how large j might be, show the same behavior with respect to the sequence of the power orders of x, x', x'', etc. This will yield the pattern shown in Fig.23.

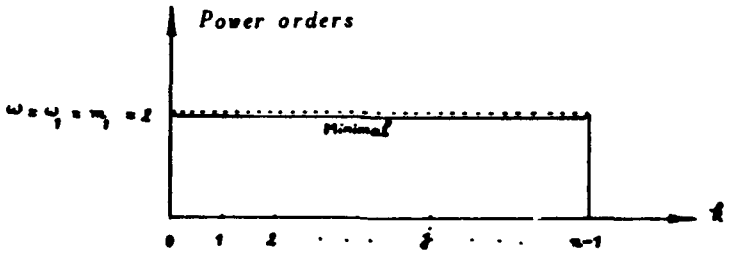


Fig.23

As a final example, let us consider the general differential equation with constant coefficients

$$L[\hat{x}] \equiv \hat{x}^{(n-j)} + a_1 \hat{x}^{(n-j-1)} + \ldots + a_{n-j} \hat{x} = f(t)$$
$$(a_{n-j} \neq 0), \tag{329}$$

where f(t) is to have the period P. It can be readily proved that all solutions, periodic with P, of the reduced homogeneous differential equation are functions with a mean value of zero. If, namely, a function $\hat{y}(t)$, periodic with P, is introduced into the homogeneous reduced differential equation, all terms which are multiplied by the derivatives $\hat{y}'$, $\hat{y}''$, ..., $\hat{y}^{(n-j)}$, have a mean value of zero. Consequently, also the last term, $a_{n-j}\hat{y}(t)$, must have a zero mean value.

It follows from the above statement that the functions /189

$$\hat{q}_{(\nu)}(t) = \hat{y}_{(\nu)}(t) \quad (\nu = 1, \ldots, \hat{\rho}) \quad \text{[see, for example, eq.(250)]}$$

have a mean value of zero. We must now prove that all functions $\hat{\varphi}_\mu(t)(\mu = (\nu) + 1, \ldots, [\nu]; \nu = 1, \ldots, \hat{\rho})$ must have a mean value of zero.

For example, the differential equation, homogeneous to eq.(329), is solved

by means of the argument $\hat{y}(t) = e^{\alpha^* t}$. This is assumed to yield the characteristic equation

$$(\alpha^* - \alpha_1^*)^{\hat{m}_1} (\alpha^* - \alpha_2^*)^{\hat{m}_2} \cdots (\alpha^* - \alpha_{\hat{s}}^*)^{\hat{m}_{\hat{s}}} = 0 \qquad (330)$$

with pairwise different values $\alpha_1^*, \ldots, \alpha_{\hat{s}}^*$ where, in addition to a complex α_μ^*, also the conjugate-complex root $\bar{\alpha}_\mu^*$ occurs. Then, the general solution of the differential equation homogeneous to eq.(329) will read

$$\hat{y}(t) = \sum_{\nu=1}^{\hat{s}} e^{\alpha_\nu^* t} P_\nu(t), \qquad (331)$$

where $P_\nu(t)$ represents an arbitrary polynomial of the degree $\hat{m}_\nu - 1$:

$$P_\nu(t) = {}^\nu c_0 \frac{t^{\hat{m}_\nu - 1}}{(\hat{m}_\nu - 1)!} + {}^\nu c_1 \frac{t^{\hat{m}_\nu - 2}}{(\hat{m}_\nu - 2)!} + \cdots + {}^\nu c_{\hat{m}_\nu - 2} t + {}^\nu c_{\hat{m}_\nu - 1}. \qquad (332)$$

The differential equation, homogeneous to eq.(329), has solutions with the period P exactly when the solutions $\alpha_1^*, \ldots, \alpha_{\hat{s}}^*$ of the characteristic equation (330) contain whole multiples of $\frac{2\pi i}{P}$. Let us assume that this is true for the values $\alpha_1^*, \ldots, \alpha_{\hat{s}}^*$ while the remaining α_μ^* are no whole multiples of $\frac{2\pi i}{P}$. We repeat here that $\alpha_\nu^* \neq 0$ since, otherwise, the differential equation homogeneous to eq.(329) would have to have a constant solution which contradicts the stipulation of $a_{n-1} \neq 0$. For the eigenvalues $\alpha_\nu^*(\nu = 1, \ldots, \hat{\rho}, \ldots, \hat{s})$, the following bound [see (Bibl.1) eq.(20)]

$$-\frac{\pi}{P} < \mathfrak{I}(\alpha_\nu^*) \leqq \frac{\pi}{P} \qquad (333)$$

at first, is not valid. Generally, we put

$$\alpha_\nu^* = \alpha_\nu + i\beta_\nu \qquad (\nu = 1, \ldots, \hat{\rho}, \ldots, \hat{s}). \qquad (334)$$

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According to eqs.(331), (332), and (334), the first row of the fundamental solution matrix $\hat{\eta}(t)$ has the form

$$\hat{\eta}^T = (\hat{\eta}_1^T, \hat{\eta}_2^T, \ldots, \hat{\eta}_{\hat{s}}^T) \qquad (335)$$

with

$$\hat{\mathfrak{y}}_{\hat{\nu}}^{T} = e^{i\beta_\nu t}\left(1, t, \ldots, \frac{t^{\hat{m}_\nu - 1}}{(\hat{m}_\nu - 1)!}\right) e^{\alpha_\nu t}$$

$$= \left(e^{i\beta_\nu t}, 0_2, \ldots, 0_{\hat{m}_\nu}\right) \cdot \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{\hat{m}_\nu - 1}}{(\hat{m}_\nu - 1)!} \\ & \ddots & & & \vdots \\ & & & & t \\ & & & & 1 \end{bmatrix} e^{\alpha_\nu t} \tag{336}$$

$$= \hat{\bar{\mathfrak{y}}}_{\nu}^{T} \cdot e^{\hat{\mathfrak{B}}_\nu t}$$

[see eq.(95)]. From this, we obtain directly

$$\hat{\varphi}_{(\nu)}(t) = e^{i\beta_\nu t}, \quad \hat{\varphi}_\mu(t) \equiv 0 \text{ for } \mu = (\nu)+1, \ldots, [\nu], (\nu = 1, \ldots, \hat{\rho}). \tag{337}$$

That $\hat{\Phi}_{(\nu)}(t)$ ($\nu = 1, \ldots, \hat{\rho}$) has a mean value of zero was determined above; naturally, the remaining $\hat{\Phi}_\mu \equiv 0$ also have a mean value of zero.

The above statements yield the following theorem:

Theorem 28: In the case of a differential equation with constant coefficients, no i_ν exists.

Consequently, the matrix $\mathfrak{A} = \mathfrak{A}^0$ automatically is in the Jordan normal form [see eqs.(114), (123)]; for the transitory transformation matrix, $\mathfrak{T} = \mathfrak{T}$ is valid.

Consequently, the scheme for the minimal orders has the following form:

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	j = o	1	2	3	j ≥ o
m_0	o	1	2	3	j
m_1	$\hat{m}_1$	$\hat{m}_1$	$\hat{m}_1$	$\hat{m}_1$	$\hat{m}_1$
m_2	$\hat{m}_2$	$\hat{m}_2$	$\hat{m}_2$	$\hat{m}_2$	$\hat{m}_2$
$m_{\hat{\rho}}$	$\hat{m}_{\hat{\rho}}$	$\hat{m}_{\hat{\rho}}$	$\hat{m}_{\hat{\rho}}$	$\hat{m}_{\hat{\rho}}$	$\hat{m}_{\hat{\rho}}$

(338)

In general, at low values of j and if the resonance case is present for the reduced differential equation, the resonance subcase cannot be of any significance for the index $\nu = 0$. However, as soon as j is sufficiently large, m_0 will

predominate over all other orders $m_1, \ldots, m_\delta$ and it becomes of importance to determine whether $\nu = 0$ is a resonance index or not. For this we need the solution $z_{[[0]]}(t)$, periodic with P, of the adjoint homogeneous differential equation, which already is defined for $j = 1$,

$$\bar{L}\left[z_{[[0]]}\right] = -\frac{d}{dt}\bar{L}\left[z_{[[0]]}\right] = 0 . \tag{339}$$

First, we will demonstrate that $z_{[[0]]}(t)$ must be constant. For this, we start from eq.(329) with f(t) of zero mean value. By j integrations and making use of the auxiliary theorem in Section 3, we obtain an equation

$$L[x] \equiv x^{(n)} + a_1 x^{(n-1)} + \ldots + a_{n-j} x^{(j)} = F(t) \tag{340}$$

with an F(t) of zero mean value. Since $\hat{x}(t)$ has the power order of

$$\hat{m} = \max_{(\nu = 1, \ldots, \hat{f})} (\hat{m}_\nu)$$

the same power order will be valid also for x(t). Since we have $m_0 = j$ in accordance with eq.(338), it follows that $\nu = 0$ is the exceptional index for $j > \hat{m}$. Consequently, it then is necessary that

$$\int_0^P z_{[[0]]}(\tau) F(\tau) d\tau = 0 \tag{341}$$

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The assumption

$$z_{[[0]]}(t) = k + \tilde{z}_{[[0]]}(t) \tag{342}$$

with $\tilde{z}_{[[0]]}$ of zero mean value and constant k would lead to a contradiction with eq.(341) if

$$F(t) = \tilde{z}_{[[0]]}(t)$$

is selected. Therefore, it is obvious that in eq.(342), we have $\tilde{z}_{[[0]]}(t) \equiv 0$.

The constant k is determined in accordance with eq.(174) as

$$z_{[[0]]}(t) = k = \frac{1}{a_{n-j}} . \tag{343}$$

From eq.(179) we then find that $\nu = 0$ is the resonance index for the differential equation (329), provided that f(t) has a mean value differing from zero and will be the exceptional index if the mean value of f(t) is zero (see also footnote on p.35).

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CURRICULUM VITAE

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I, Rahmi Ibrahim Ibrahim Abdel Karim, was born on 2 February 1932 as the son of the merchant J.J.Abdel Karim and his wife Seneb, in Tanta Egypt.

In Tanta, I visited elementary school and then junior high school until 1943, followed by the Tanta high school up to graduation in 1948.

Between 1948 and 1952, I attended the Natural-Science Department of Schams University in Cairo, studying mathematics and natural sciences. In 1952, I passed the exam for the degree of Bachelor of Science (BS). Later, I became mathematics teacher for the upper grades of the Giza High School and then at the Orman-Model High School in Cairo/Giza.

In May 1956, I passed the exam in mathematics for the diploma of the Institute for Higher Studies for Teachers in the Mathematical Department for Higher Education of the Ministry of Education and Schools in Cairo. In December 1956, I received the "special diploma" from the College of Education A' at Schams University, Cairo.

Since beginning of 1959, I am holding a scholarship by the Allied Arab Republic for further scientific studies in the field of mathematics in the Federal Republic of Germany.

In spring 1959, I successfully completed a course in German at the Goethe-Institute in Blaubeuren near Ulm. After this, I studied mathematics, applied mathematics, and mechanics during the summer semester 1959 at the Braunschweig Polytechnic Institute. From March 1960 to date, I worked on my thesis. This thesis has been published in a journal in the three following parts:

1) On the resonance concept in systems of n ordinary linear differ- /195

ential equations of the first order;

2) On the resonance case in systems of n ordinary nonlinear differential equations of the first order;

3) Study of the resonance case in systems of ordinary linear differential equations.

The fourth and last part:

4) The resonance case in ordinary linear differential equations of the n^{th} order

will be published, in part, in a journal.

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